

ALMOST RING THEORY

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1. INTRODUCTION

Broadly speaking, the aim of this work is to describe “how to do ring theory” within monoidal categories that arise as localisations of categories of modules over certain rings. A reader looking for forerunners of our themes would be drawn inevitably to Gabriel’s “Des catégories abéliennes” [8], and might even conclude that Gabriel’s memoir must have been the main instigation for the present article. In truth, the initial motivation is to be found elsewhere, namely in the want of adequately documented foundations for the method of almost étale extensions that underpins Faltings’ approach to p -adic Hodge theory as presented in [6]. However, as is often the case with healthy offsprings, our subject matter has eventually resolved to venture beyond its original boundaries and pursue an autonomous existence.

In any case, we are glad to report that our paper remains true to its first vocation, which is to serve as a comprehensive reference, paving the way to deeper aspects of almost étale theory, especially to the difficult purity theorem of [6]. The notions of almost unramified and almost étale morphism are defined and their main properties are established, including the analogues of the classical lifting theorems over nilpotent extensions, and invariance under Frobenius. Also, to any almost finitely presented almost flat morphism we attach an almost trace form, and we characterize almost étale morphisms in terms of this form. Finally, we study some cases of non-flat descent for almost étale maps.

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Actually, our terminology is slightly different, in that we replace usual modules and algebras by their “almost” counterparts, which live in the category of almost modules, a localisation of the category of modules. So for instance, instead of almost étale morphisms of algebras, we have étale morphisms of almost algebras.

The categories of almost modules (or almost algebras) and of usual modules (resp. algebras) are linked in manifold ways. First of all we have of course the localisation functor. Then, as it had already been observed by Gabriel, there is a right adjoint to localisation. Furthermore, we show that there is a *left* adjoint as well, that, to our knowledge, has not been exploited before, in spite of its several useful qualities which will establish it quickly as one of our main tools. The ensemble of localisation and right, left adjoints exhibits some remarkable exactness properties, that are typically associated to open imbeddings of topoi, all of which seems to suggest the existence of some deeper geometric structure, still to be unearthed. We may have encountered here an instance of a general principle, apparently evoked first by Deligne, according to which one should try to do algebraic geometry on arbitrary abelian tensor categories (notice though, that our categories are more general than the tannakian categories of [4]).

A large part of the paper is devoted to the construction and study of the almost version of Illusie’s cotangent complex, on which we base our deformation theory for almost algebras. Faltings’ original method was based on Hochschild cohomology rather than the cotangent complex. While Faltings’ approach has the advantage of being more explicit and elementary, it also has the drawback of involving a very large number of long and tedious manipulations with cocycles, and requires a painstaking tracking of the “epsilon book-keeping”. The method presented here avoids (or at least removes from view!) these problems, and also leads to more general results (especially, we can drop all finiteness assumptions from the statements of the lifting theorems).

Though we have strived throughout for the widest generality, in a few places one could have gone even further : for instance it would have been possible to globalise all definitions and most results to arbitrary schemes. However, the extension to schemes is completely straightforward, and in practice seems to be scarcely useful. Similarly, there is currently not much incentive to study a notion of “almost smooth morphism”.

2. HOMOLOGICAL THEORY

2.1. Some ring-theoretic preliminaries. Unless otherwise stated, every ring is commutative with unit. Our basic setup consists of a fixed base ring V containing an ideal \mathfrak{m} such that $\mathfrak{m}^2 = \mathfrak{m}$. Starting from section 2.4, we will also assume that $\tilde{\mathfrak{m}} = \mathfrak{m} \otimes_V \mathfrak{m}$ is a flat V -module.

Example 2.1.1. i) The main example is given by a non-discrete valuation ring V with valuation $\nu : V - \{0\} \rightarrow \Gamma$ of rank one (where Γ is the totally ordered abelian group of values of ν). Then we can take $\mathfrak{m} = \{0\} \cup \{x \in V - \{0\} \mid \nu(x) > \nu(1)\}$.

ii) Suppose that $\mathfrak{m} = V$. This is the “classical limit”. In this case almost ring theory reduces to usual ring theory. Thus, all the discussion that follows specialises to, and sometimes gives alternative proofs for, statements about rings and their modules.

We define a uniform structure on the set \mathcal{I} of ideals of V as follows. For every finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ the subset of $\mathcal{I} \times \mathcal{I}$ given by $\{(I, J) \mid \mathfrak{m}_0 \cdot J \subset I \text{ and } \mathfrak{m}_0 \cdot I \subset J\}$ is an entourage for the uniform structure, and the subsets of this kind form a fundamental system of entourages (cp. [3] Ch.II §1). The uniform structure induces a topology on \mathcal{I} and moreover the notion of convergent (resp. Cauchy) sequence of ideals is well defined. We will also need a notion of “Cauchy product” : let $\prod_{n=0}^{\infty} I_n$ be a formal infinite product of ideals. We say that the formal product *satisfies the Cauchy condition* (or briefly : *is a Cauchy product*) if, for every neighborhood \mathcal{U} of $V \in \mathcal{I}$ there exists $n_0 \geq 0$ such that $\prod_{m=n}^{n+p} I_m \in \mathcal{U}$ for all $n \geq n_0$ and all $p \geq 0$.

Remark 2.1.2. Suppose that $J_0 \subset J_1 \subset \dots$ is an increasing infinite sequence of ideals of I such that $\lim_{k \rightarrow \infty} J_k = V$ (convergence for the above uniform structure on \mathcal{A}). Then one checks easily that $\bigcup_{k=0}^{\infty} \mathfrak{m} \cdot J_k = \mathfrak{m}$.

Let M be a given V -module. We say that M is *almost zero* if $\mathfrak{m} \cdot M = 0$. A map ϕ of V -modules is an *almost isomorphism* if both $\text{Ker}(\phi)$ and $\text{Coker}(\phi)$ are almost zero V -modules.

Remark 2.1.3. (i) It is easy to check that a V -module M is almost zero if and only if $\mathfrak{m} \otimes_V M = 0$. Similarly, a map $M \rightarrow N$ of V -modules is an almost isomorphism if and only if the induced map $\tilde{\mathfrak{m}} \otimes_V M \rightarrow \tilde{\mathfrak{m}} \otimes_V N$ is an isomorphism. Notice also that, if \mathfrak{m} is flat, then $\mathfrak{m} \simeq \tilde{\mathfrak{m}}$.

(ii) Let $V \rightarrow W$ be a ring homomorphism. For a V -module M set $M_W = W \otimes_V M$. We have an exact sequence

$$(2.1.4) \quad 0 \rightarrow K \rightarrow \mathfrak{m}_W \rightarrow \mathfrak{m} \cdot W \rightarrow 0$$

where $K = \text{Tor}_1^V(V/\mathfrak{m}, W)$ is an almost zero W -module. By (i) it follows that $\mathfrak{m} \otimes_V K \simeq (\mathfrak{m} \cdot W) \otimes_W K \simeq 0$. Then, applying $\mathfrak{m}_W \otimes_W -$ and $- \otimes_W (\mathfrak{m} \cdot W)$ to (2.1.4) we derive

$$\mathfrak{m}_W \otimes_W \mathfrak{m}_W \simeq \mathfrak{m}_W \otimes_W (\mathfrak{m} \cdot W) \simeq (\mathfrak{m} \cdot W) \otimes_W (\mathfrak{m} \cdot W)$$

i.e. $\tilde{\mathfrak{m}}_W \simeq (\mathfrak{m} \cdot W)^\sim$. In particular, if $\tilde{\mathfrak{m}}$ is a flat V -module, then $\tilde{\mathfrak{m}}_W$ is a flat W -module. This means that our basic assumptions on the pair (V, \mathfrak{m}) are stable under arbitrary base extension. Notice that the flatness of \mathfrak{m} does not imply the flatness of $\mathfrak{m} \cdot W$. This partly explains why we insist that $\tilde{\mathfrak{m}}$, rather than \mathfrak{m} , be flat.

Before moving on, we want to analyze in some detail how our basic assumptions relate to certain other natural conditions that can be postulated on the pair (V, \mathfrak{m}) . Indeed, let us consider the following two hypotheses :

(A) $\mathfrak{m} = \mathfrak{m}^2$ and \mathfrak{m} is a filtered colimit of principal ideals.

(B) $\mathfrak{m} = \mathfrak{m}^2$ and, for all integers $k > 1$, the k -th powers of elements of \mathfrak{m} generate \mathfrak{m} .

Clearly (A) implies (B). Less obvious is the following result.

Proposition 2.1.5. (i) (A) implies that $\tilde{\mathfrak{m}}$ is flat.

(ii) If $\tilde{\mathfrak{m}}$ is flat then (B) holds.

Proof. Suppose that (A) holds, so that $\mathfrak{m} = \text{colim}_{\alpha \in I} Vx_\alpha$, where I is a directed set parametrizing elements $x_\alpha \in \mathfrak{m}$ (and $\alpha \leq \beta \Leftrightarrow Vx_\alpha \subset Vx_\beta$). For any $\alpha \in I$ we have natural isomorphisms

$$(2.1.6) \quad Vx_\alpha \simeq V/\text{Ann}_V(x_\alpha) \simeq (Vx_\alpha) \otimes_V (Vx_\alpha).$$

For $\alpha \leq \beta$, let $j_{\alpha\beta} : Vx_\alpha \hookrightarrow Vx_\beta$ be the imbedding; we have a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\mu_{z^2}} & V \\ \pi_\alpha \downarrow & & \downarrow \pi_\beta \\ (Vx_\alpha) \otimes_V (Vx_\alpha) & \xrightarrow{j_{\alpha\beta} \otimes j_{\alpha\beta}} & (Vx_\beta) \otimes_V (Vx_\beta) \end{array}$$

where $z \in V$ is such that $x_\alpha = z \cdot x_\beta$, μ_{z^2} is multiplication by z^2 and π_α is the projection induced by (2.1.6) (and similarly for π_β). Since $\mathfrak{m} = \mathfrak{m}^2$, for all $\alpha \in I$ we can find β such that x_α is a multiple of x_β^2 . Say $x_\alpha = y \cdot x_\beta^2$; then we can take $z = y \cdot x_\beta$, so z^2 is a multiple of x_α and in the above diagram $\text{Ker}(\pi_\alpha) \subset \text{Ker}(\mu_{z^2})$. Hence one can define a map $\lambda_{\alpha\beta} : (Vx_\alpha) \otimes_V (Vx_\alpha) \rightarrow V$ such that $\pi_\beta \circ \lambda_{\alpha\beta} = j_{\alpha\beta} \otimes j_{\alpha\beta}$ and $\lambda_{\alpha\beta} \circ \pi_\alpha = \mu_{z^2}$. It now follows that for every V -module N , the induced morphism $\text{Tor}_1^V(N, (Vx_\alpha) \otimes_V (Vx_\alpha)) \rightarrow \text{Tor}_1^V(N, (Vx_\beta) \otimes_V (Vx_\beta))$ is the zero map. Taking the colimit we derive that $\tilde{\mathfrak{m}}$ is flat. This shows (i). In order to show (ii) we consider, for any prime number p , the following condition

(\ast_p) $\mathfrak{m}/p \cdot \mathfrak{m}$ is generated (as a V -module) by the p -th powers of its elements.

Clearly **(B)** implies (\ast_p) for all p . In fact we have :

Claim 2.1.7. **(B)** holds if and only if (\ast_p) holds for every prime p .

Proof of the claim: Suppose that (\ast_p) holds for every prime p . The polarization identity

$$k! \cdot x_1 \cdot x_2 \cdot \dots \cdot x_k = \sum_{I \subset \{1,2,\dots,k\}} (-1)^{k-|I|} \cdot \left(\sum_{i \in I} x_i \right)^k$$

shows that if $N = \sum_{x \in \mathfrak{m}} Vx^k$ then $k! \cdot \mathfrak{m} \subset N$. To prove that $N = \mathfrak{m}$ it then suffices to show that for every prime p dividing $k!$ we have $\mathfrak{m} = p \cdot \mathfrak{m} + N$. Let $\phi : V/p \cdot V \rightarrow V/p \cdot V$ be the Frobenius ($x \mapsto x^p$); we can denote by $(V/p \cdot V)^\phi$ the ring $V/p \cdot V$ seen as a $V/p \cdot V$ -algebra via the homomorphism ϕ . Also set $\phi^* M = M \otimes_{V/p \cdot V} (V/p \cdot V)^\phi$ for a $V/p \cdot V$ -module M . Then the map $\phi^*(\mathfrak{m}/p \cdot \mathfrak{m}) \rightarrow (\mathfrak{m}/p \cdot \mathfrak{m})$ (defined by raising to p -th power) is surjective by (\ast_p). Hence so is $(\phi^r)^*(\mathfrak{m}/p \cdot \mathfrak{m}) \rightarrow (\mathfrak{m}/p \cdot \mathfrak{m})$ for every $r > 0$, which says that $\mathfrak{m} = p \cdot \mathfrak{m} + N$ when $k = p^r$, hence for every k .

Next recall (see [11] Exp. XVII 5.5.2) that, if M is a V -module, the module of symmetric tensors $\text{TS}^k(M)$ is defined as $(\otimes_V^k M)^{S_k}$, the invariants under the natural action of the symmetric group S_k on $\otimes_V^k M$. We have a natural map $\Gamma^k(M) \rightarrow \text{TS}^k(M)$ that is an isomorphism when M is flat (see *loc. cit.* 5.5.2.5; here Γ^k denotes the k -th graded piece of the divided power algebra).

Claim 2.1.8. The group S_k acts trivially on $\otimes_V^k \tilde{\mathfrak{m}}$ and the map $\tilde{\mathfrak{m}} \otimes_V \mathfrak{m} \rightarrow \tilde{\mathfrak{m}} (x \otimes y \otimes z \mapsto x \otimes yz)$ is an isomorphism.

Proof of the claim: The first statement is reduced to the case of transpositions and to $k = 2$. There we can compute : $x \otimes yz = xy \otimes z = y \otimes xz = yz \otimes x$. For the second statement note that the imbedding $\mathfrak{m} \hookrightarrow V$ is an almost isomorphism, and apply remark 2.1.3(i).

Suppose now that $\tilde{\mathfrak{m}}$ is flat and pick a prime p . Then S_p acts trivially on $\otimes_V^p \tilde{\mathfrak{m}}$. Hence

$$(2.1.9) \quad \Gamma^p(\tilde{\mathfrak{m}}) \simeq \otimes_V^p \tilde{\mathfrak{m}} \simeq \tilde{\mathfrak{m}}.$$

But $\Gamma^p(\tilde{\mathfrak{m}})$ is spanned as a V -module by the products $\gamma_{i_1}(x_1) \cdot \dots \cdot \gamma_{i_k}(x_k)$ (where $x_i \in \tilde{\mathfrak{m}}$ and $\sum_j i_j = p$). Under the isomorphism (2.1.9) these elements map to $\binom{p}{i_1, \dots, i_k} \cdot x_1^{i_1} \cdot \dots \cdot x_k^{i_k}$; but such an element vanishes in $\tilde{\mathfrak{m}}/p \cdot \tilde{\mathfrak{m}}$ unless $i_k = p$ for some k . Therefore $\tilde{\mathfrak{m}}/p \cdot \tilde{\mathfrak{m}}$ is generated by p -th powers, so the same is true for $\mathfrak{m}/p \cdot \mathfrak{m}$, and by the above, **(B)** holds, which shows (ii). \square

Proposition 2.1.10. Suppose that \mathfrak{m} is countably generated as a V -module. Then we have :

- i) $\tilde{\mathfrak{m}}$ is countably presented as a V -module;
- ii) if $\tilde{\mathfrak{m}}$ is a flat V -module, then it is of homological dimension ≤ 1 .

Proof. Let $(\varepsilon_i)_{i \in I}$ be a countable generating family of \mathfrak{m} . Then $\varepsilon_i \otimes \varepsilon_j$ generate $\tilde{\mathfrak{m}}$ and $\varepsilon_i \cdot \varepsilon_j \cdot (\varepsilon_k \otimes \varepsilon_l) = \varepsilon_k \cdot \varepsilon_l \cdot (\varepsilon_i \otimes \varepsilon_j)$ for all $i, j, k, l \in I$. For every $i \in I$, we can write $\varepsilon_i = \sum_j x_{ij} \varepsilon_j$, for certain $x_{ij} \in \mathfrak{m}$. Let F be the V -module defined by generators $(e_{ij})_{i,j \in I}$, subject to the relations:

$$\varepsilon_i \cdot \varepsilon_j \cdot e_{kl} = \varepsilon_k \cdot \varepsilon_l \cdot e_{ij} \quad e_{ik} = \sum_j x_{ij} e_{jk} \quad \text{for all } i, j, k, l \in I.$$

We get an epimorphism $\pi : F \rightarrow \tilde{\mathfrak{m}}$ by $e_{ij} \mapsto \varepsilon_i \otimes \varepsilon_j$. The relations imply that, if $x = \sum_{k,l} y_{kl} \in \text{Ker}(\pi)$, then $\varepsilon_i \cdot \varepsilon_j \cdot x = 0$, so $\mathfrak{m} \cdot \text{Ker}(\pi) = 0$. Whence $\mathfrak{m} \otimes_V \text{Ker}(\pi) = 0$ and $\mathbf{1}_{\mathfrak{m}} \otimes_V \pi$ is an isomorphism. We consider the diagram

$$\begin{array}{ccc} \mathfrak{m} \otimes_V F & \xrightarrow{\sim} & \mathfrak{m} \otimes_V \tilde{\mathfrak{m}} \\ \phi \downarrow & & \downarrow \psi \\ F & \xrightarrow{\pi} & \tilde{\mathfrak{m}} \end{array}$$

where ϕ and ψ are induced by scalar multiplication. We already know that ψ is an isomorphism, and since $F = m \cdot F$, we see that ϕ is an epimorphism, so π is an isomorphism, which shows (i). Now (ii) follows from (i), since it is well-known that a flat countably presented module is of homological dimension ≤ 1 (see [16] (Ch.I, Th.3.2) and the discussion in [19] pp.49-50). \square

2.2. Almost categories. If \mathcal{C} is a category, and X, Y two objects of \mathcal{C} , we will usually denote by $\text{Hom}_{\mathcal{C}}(X, Y)$ the set of morphisms in \mathcal{C} from X to Y and by $\mathbf{1}_X$ the identity morphism of X . Moreover we denote by \mathcal{C}^o the opposite category of \mathcal{C} and by $s.\mathcal{C}$ the category of simplicial objects over \mathcal{C} , that is, functors $\Delta^o \rightarrow \mathcal{C}$, where Δ is the category whose objects are the ordered sets $[n] = \{0, \dots, n\}$ for each integer $n \geq 0$ and where a morphism $\phi : [p] \rightarrow [q]$ is a non-decreasing map. A morphism $f : X \rightarrow Y$ in $s.\mathcal{C}$ is a sequence of morphisms $f_{[n]} : X[n] \rightarrow Y[n]$, $n \geq 0$ such that the obvious diagrams commute. We can imbed \mathcal{C} in $s.\mathcal{C}$ by sending each object X to the “constant” object $s.X$ such that $s.X[n] = X$ for all $n \geq 0$ and $s.X[\phi] = \mathbf{1}_X$ for all morphisms ϕ in Δ .

If \mathcal{C} is an abelian category, $\mathbf{D}(\mathcal{C})$ will denote the derived category of \mathcal{C} . As usual we have also the full subcategories $\mathbf{D}^+(\mathcal{C}), \mathbf{D}^-(\mathcal{C})$ of complexes of objects of \mathcal{C} that are exact for sufficiently large negative (resp. positive) degree. If R is a ring, the category of R -modules (resp. R -algebras) will be denoted by $R\text{-Mod}$ (resp. $R\text{-Alg}$). Most of the times we will write $\text{Hom}_R(M, N)$ instead of $\text{Hom}_{R\text{-Mod}}(M, N)$.

We denote by Set the category of sets. The symbol \mathbb{N} denotes the set of non-negative integers; in particular $0 \in \mathbb{N}$.

The full subcategory Σ of $V\text{-Mod}$ consisting of all V -modules that are almost isomorphic to 0 is clearly a Serre subcategory and hence we can form the quotient category $V\text{-Mod}/\Sigma$. There is a localization functor

$$V\text{-Mod} \rightarrow V\text{-Mod}/\Sigma \quad M \mapsto M^a$$

that takes a V -module M to the same module, seen as an object of $V\text{-Mod}/\Sigma$. In particular, we have the object V^a associated to V ; it seems therefore natural to use the notation $V^a\text{-Mod}$ for the category $V\text{-Mod}/\Sigma$, and an object of $V^a\text{-Mod}$ will be indifferently referred to as “a V^a -module” or “an almost V -module”. In case we need to stress the dependance on the ideal m , we can write $(V, m)^a\text{-Mod}$.

Since the almost isomorphisms form a multiplicative system (see *e.g.* [22] Exerc.10.3.2), it is possible to describe the morphisms in $V^a\text{-Mod}$ via a calculus of fractions, as follows. Let $V\text{-al.Iso}$ be the category that has the same objects as $V\text{-Mod}$, but such that $\text{Hom}_{V\text{-al.Iso}}(M, N)$ consists of all almost isomorphisms $M \rightarrow N$. If M is any object of $V\text{-al.Iso}$ we write $(V\text{-al.Iso}/M)$ for the category of objects of $V\text{-al.Iso}$ over M (*i.e.* morphisms $\phi : X \rightarrow M$). If $\phi_i : X_i \rightarrow M$ ($i = 1, 2$) are two objects of $(V\text{-al.Iso}/M)$ then $\text{Hom}_{(V\text{-al.Iso}/M)}(\phi_1, \phi_2)$ consists of all morphisms $\psi : X_1 \rightarrow X_2$ in $V\text{-al.Iso}$ such that $\phi_1 = \phi_2 \circ \psi$. For any two V -modules M, N we define a functor $\mathcal{F}_N : (V\text{-al.Iso}/M)^o \rightarrow V\text{-Mod}$ by associating to an object $\phi : P \rightarrow M$ the V -module $\text{Hom}_V(P, N)$ and to a morphism $\alpha : P \rightarrow Q$ the map $\text{Hom}_V(Q, N) \rightarrow \text{Hom}_V(P, N) : \beta \mapsto \beta \circ \alpha$. Then we have

$$(2.2.1) \quad \text{Hom}_{V^a\text{-Mod}}(M^a, N^a) = \text{colim}_{(V\text{-al.Iso}/M)^o} \mathcal{F}_N.$$

However, formula (2.2.1) can be simplified considerably, by remarking that, for any V -module M , the natural morphism $\tilde{m} \otimes_V M \rightarrow M$ is an initial object of $(V\text{-al.Iso}/M)$. Indeed, let $\phi : N \rightarrow M$ be an almost isomorphism; the diagram

$$\begin{array}{ccc} \tilde{m} \otimes_V N & \xrightarrow{\sim} & \tilde{m} \otimes_V M \\ \downarrow & & \downarrow \\ N & \xrightarrow{\phi} & M \end{array}$$

(cp. remark 2.1.3(i)) allows one to define a morphism $\psi : \tilde{\mathfrak{m}} \otimes_V M \rightarrow N$ over M . We need to show that ψ is unique. But if $\psi_1, \psi_2 : \tilde{\mathfrak{m}} \otimes_V M \rightarrow N$ are two maps over M , then $\text{Im}(\psi_1 - \psi_2) \subset \text{Ker}(\phi)$ is almost zero, hence $\text{Im}(\psi_1 - \psi_2) = 0$, since $\tilde{\mathfrak{m}} \otimes_V M = \mathfrak{m} \cdot (\tilde{\mathfrak{m}} \otimes_V M)$. Consequently, (2.2.1) boils down to

$$(2.2.2) \quad \text{Hom}_{V^a\text{-Mod}}(M^a, N^a) = \text{Hom}_V(\tilde{\mathfrak{m}} \otimes_V M, N).$$

In particular $\text{Hom}_{V^a\text{-Mod}}(M, N)$ has a natural structure of V -module for any two V^a -modules M, N , i.e. $\text{Hom}_{V^a\text{-Mod}}(-, -)$ is a bifunctor that takes values in the category $V\text{-Mod}$.

One checks easily (for instance using (2.2.2)) that the usual tensor product induces a bifunctor $- \otimes_V -$ on almost V -modules, which, in the jargon of [4] makes of $V^a\text{-Mod}$ an *abelian tensor category*. Then an *almost V -algebra* is just a commutative unitary monoid in the tensor category $V^a\text{-Mod}$. Let us recall what this means. Quite generally, let $(\mathcal{C}, \otimes, U)$ be any abelian tensor category, so that $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is a biadditive functor, U is the identity object of \mathcal{C} (see [4] p.105) and for any two objects M and N in \mathcal{C} we have a “commutativity constraint” (i.e. a functorial isomorphism $\eta_{M|N} : M \otimes N \rightarrow N \otimes M$ that “switches the two factors”) and a functorial isomorphism $\nu_M : U \otimes M \rightarrow M$. Then a \mathcal{C} -monoid A is an object of \mathcal{C} endowed with a morphism $\mu_A : A \otimes A \rightarrow A$ (the “multiplication” of A) satisfying the associativity condition

$$\mu_A \circ (\mathbf{1}_A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes \mathbf{1}_A).$$

We say that A is *unitary* if additionally A is endowed with a “unit morphism” $\mathbf{1}_A : U \rightarrow A$ satisfying the (left and right) unit property :

$$\mu_A \circ (\mathbf{1}_A \otimes \mathbf{1}_A) = \nu_A \quad \mu_A \circ (\mathbf{1}_A \otimes \mathbf{1}_A) \circ \eta_{A|U} = \mu_A \circ (\mathbf{1}_A \otimes \mathbf{1}_A).$$

Finally A is *commutative* if $\mu_A = \mu_A \circ \eta_{A|A}$ (to be rigorous, in all of the above one should indicate the associativity constraints, which we have omitted : see [4]). A commutative unitary monoid will also be simply called an *algebra*. With the morphisms defined in the obvious way, the \mathcal{C} -monoids form a category; furthermore, given a \mathcal{C} -monoid A , a *left A -module* is an object M of \mathcal{C} endowed with a morphism $\sigma_M : A \otimes M \rightarrow M$ such that $\sigma_M \circ (\mathbf{1}_A \otimes \sigma_M) = \sigma_M \circ (\mu_A \otimes \mathbf{1}_M)$. Similarly one defines right A -modules and A -bimodules. In the case of bimodules we have left and right morphisms $\sigma_{M,l} : A \otimes M \rightarrow M$, $\sigma_{M,r} : M \otimes A \rightarrow M$ and one imposes that they “commute”, i.e. that

$$\sigma_{M,r} \circ (\sigma_{M,l} \otimes \mathbf{1}_A) = \sigma_{M,l} \circ (\mathbf{1}_A \otimes \sigma_{M,r}).$$

Clearly the (left resp. right) A -modules (and the A -bimodules) form an additive category with *A -linear morphisms* defined as one expects. One defines the notion of a submodule as an equivalence class of monomorphisms $N \rightarrow M$ such that the composition $A \otimes N \rightarrow A \otimes M \rightarrow M$ factors through N . Now, if $f : M \rightarrow N$ is a morphism of left A -modules, then $\text{Ker}(f)$ exists in the underlying abelian category \mathcal{C} and one checks easily that it has a unique structure of left A -module which makes it a submodule of M . If moreover \otimes is right exact when either argument is fixed, then also $\text{Coker}(f)$ has a unique A -module structure for which $N \rightarrow \text{Coker}(f)$ is A -linear. In this case the category of left A -modules is abelian. Similarly, if A is a unitary \mathcal{C} -monoid, then one defines the notion of *unitary left A -module* by requiring that $\sigma_M \circ (\mathbf{1}_A \otimes \mathbf{1}_M) = \nu_M$ and these form an abelian category when \otimes is right exact.

Specialising to our case we obtain the category $V^a\text{-Alg}$ of almost V -algebras and, for every almost V -algebra A , the category $A\text{-Mod}$ of unitary left A -modules. Clearly the localization functor restricts to a functor $V\text{-Alg} \rightarrow V^a\text{-Alg}$ and for any V -algebra R we have a localization functor $R\text{-Mod} \rightarrow R^a\text{-Mod}$.

Next, if A is an almost V -algebra, we can define the category $A\text{-Alg}$ of A -algebras. It consists of all the morphisms $A \rightarrow B$ of almost V -algebras.

Let again $(\mathcal{C}, \otimes, U)$ be any abelian tensor category. By [4] p.119, the endomorphism ring $\text{End}_{\mathcal{C}}(U)$ of U is commutative. For any object M of \mathcal{C} , denote $M_* = \text{Hom}_{\mathcal{C}}(U, M)$; then $M \mapsto$

M_* defines a functor $\mathcal{C} \rightarrow \text{End}_{\mathcal{C}}(U)\text{-Mod}$. Moreover, if A is a \mathcal{C} -monoid, A_* is an associative $\text{End}_{\mathcal{C}}(U)$ -algebra, with multiplication given as follows. For $a, b \in A_*$ let $a \cdot b = \mu_A \circ (a \otimes b) \circ \nu_U^{-1}$. Similarly, if M is an A -module, M_* is an A_* -module in a natural way, and in this way we obtain a functor from A -modules and A -linear morphisms to A_* -modules and A_* -linear maps. Using [4] (Prop. 1.3), one can also check that $\text{End}_{\mathcal{C}}(U) = U_*$ as $\text{End}_{\mathcal{C}}(U)$ -algebras, where U is viewed as a \mathcal{C} -monoid using ν_U .

All this applies especially to our categories of almost modules and almost algebras : in this case we call $M \mapsto M_*$ the *functor of almost elements*. So, if M is an almost module, an almost element of M is just an honest element of M_* . Using (2.2.2) one can show easily that for every V -module M the natural map $M \rightarrow (M^a)_*$ is an almost isomorphism.

Let A be an almost V -algebra. For any two A -modules M, N , the set $\text{Hom}_{A\text{-Mod}}(M, N)$ has a natural structure of A_* -module and we obtain an internal Hom functor by letting

$$\text{alHom}_A(M, N) = \text{Hom}_{A\text{-Mod}}(M, N)^a.$$

This is the functor of *almost homomorphisms* from M to N .

For any A -module M we have also a functor of tensor product $M \otimes_A -$ on A -modules which, in view of the following proposition 2.2.4 can be shown to be a left adjoint to the functor $\text{alHom}_A(M, -)$. It can be defined as $M \otimes_A N = (M_* \otimes_{A_*} N_*)^a$ but an appropriate almost version of the usual construction would also work.

With this tensor product, $A\text{-Mod}$ is an abelian tensor category as well, and $A\text{-Alg}$ could also be described as the category of $(A\text{-Mod})$ -algebras. Under this equivalence, a morphism $\phi : A \rightarrow B$ of almost V -algebras becomes the unit morphism $\underline{1}_B : A \rightarrow B$ of the corresponding monoid. We will sometimes drop the subscript and write simply $\underline{1}$.

Remark 2.2.3. Let $V \rightarrow W$ be a map of base rings, W taken with the extended ideal $\mathfrak{m} \cdot W$. Then W^a is an almost V -algebra so we have defined the category $W^a\text{-Mod}$ using base ring V and the category $(W, \mathfrak{m} \cdot W)^a\text{-Mod}$ using base W . One shows easily that they are equivalent: we have an obvious functor $(W, \mathfrak{m} \cdot W)^a\text{-Mod} \rightarrow W^a\text{-Mod}$ and an essential inverse is provided by $M \mapsto M_*$. Similar base comparison statements hold for the categories of almost algebras.

Proposition 2.2.4. *i) There is a natural isomorphism $A \simeq A_*^a$ of almost V -algebras.*

ii) Let R be any V -algebra. Then the functor $M \mapsto M_$ from $R^a\text{-Mod}$ to $R\text{-Mod}$ (resp. from $R^a\text{-Alg}$ to $R\text{-Alg}$) is right adjoint to the localization functor $R\text{-Mod} \rightarrow R^a\text{-Mod}$ (resp. $R\text{-Alg} \rightarrow R^a\text{-Alg}$).*

iii) The counit of the adjunction $M_^a \rightarrow M$ is a natural isomorphism from the composition of the two functors to the identity functor $\mathbf{1}_{A\text{-Mod}}$ (resp. $\mathbf{1}_{A\text{-Alg}}$).*

Proof. (i) has already been remarked. We show (ii). In light of remark 2.2.3 (applied with $W = R$) we can assume that $V = R$. Let M be a V -module and N an almost V -module; we have natural bijections

$$\begin{aligned} \text{Hom}_{V^a\text{-Mod}}(M^a, N) &\simeq \text{Hom}_{V^a\text{-Mod}}(M^a, (N_*)^a) \simeq \text{Hom}_V(\tilde{\mathfrak{m}} \otimes_V M, N_*) \\ &\simeq \text{Hom}_V(M, \text{Hom}_V(\tilde{\mathfrak{m}}, N_*)) \simeq \text{Hom}_V(M, \text{Hom}_{V^a\text{-Mod}}(V, (N_*)^a)) \\ &\simeq \text{Hom}_V(M, N_*) \end{aligned}$$

which proves (ii). Now (iii) follows by inspecting the proof of (ii), or by [8] (ch.III Prop.3). \square

Remark 2.2.5. The existence of the right adjoint follows also directly from [8] (chap.III §3 Cor.1 or chap.V §2).

Corollary 2.2.6. *The functor $M \mapsto M_*$ from $R^a\text{-Mod}$ to $R\text{-Mod}$ sends injectives to injectives and injective envelopes to injective envelopes.*

Proof. The functor $M \mapsto M_*$ is right adjoint to an exact functor, hence it preserves injectives. Now, let J be an injective envelope of M ; to show that J_* is an injective envelope of M_* , it

suffices to show that J_* is an essential extension of M_* . However, if $N \subset J_*$ and $N \cap M_* = 0$, then $N^a \cap M = 0$, hence $\mathfrak{m} \cdot N = 0$, but J_* does not contain \mathfrak{m} -torsion, thus $N = 0$. \square

Corollary 2.2.7. *The categories $A\text{-Mod}$ and $A\text{-Alg}$ are both complete and cocomplete.*

Proof. We recall that the categories $A_*\text{-Mod}$ and $A_*\text{-Alg}$ are both complete and cocomplete. Now let I be any small indexing category and $M : I \rightarrow A\text{-Mod}$ be any functor. Denote by $M_* : I \rightarrow A_*\text{-Mod}$ the composed functor $i \mapsto M(i)_*$. We claim that $\text{colim}_I M = (\text{colim}_I M_*)^a$. The proof is an easy application of proposition 2.2.4(iii). A similar argument also works for limits and for the category $A\text{-Alg}$. \square

Note that the essential image of $M \mapsto M_*$ is closed under limits. Next recall that the forgetful functor $A_*\text{-Alg} \rightarrow \text{Set}$ (resp. $A_*\text{-Mod} \rightarrow \text{Set}$) has a left adjoint $A_*[-] : \text{Set} \rightarrow A_*\text{-Alg}$ (resp. $A^{(-)} : \text{Set} \rightarrow A_*\text{-Mod}$) that assigns to a set S the free A_* -algebra $A_*[S]$ (resp. the free A_* -module $A_*^{(S)}$) generated by S . If S is any set, it is natural to write $A[S]$ (resp. $A^{(S)}$) for the A -algebra $(A_*[S])^a$ (resp. for the A -module $(A_*^{(S)})^a$). This yields a left adjoint, called the *free A -algebra* functor $\text{Set} \rightarrow A\text{-Alg}$ (resp. the *free A -module* functor $\text{Set} \rightarrow A\text{-Mod}$) to the “forgetful” functor $A\text{-Alg} \rightarrow \text{Set}$ (resp. $A\text{-Mod} \rightarrow \text{Set}$) $B \mapsto B_*$.

Now let R be any V -algebra; we want to construct a left adjoint to the localisation functor $R\text{-Mod} \rightarrow R^a\text{-Mod}$. For a given R^a -module M , let

$$(2.2.8) \quad M_{\dagger} = \tilde{\mathfrak{m}} \otimes_V (M_*).$$

We have the natural map (unit of adjunction) $R \rightarrow R_*$, so that we can view M_{\dagger} as an R -module.

Proposition 2.2.9. *i) The functor $R^a\text{-Mod} \rightarrow R\text{-Mod}$ defined by (2.2.8) is left adjoint to localisation.*

ii) The unit of the adjunction $M \rightarrow M_{\dagger}$ is a natural isomorphism from the identity functor $1_{R^a\text{-Mod}}$ to the composition of the two functors.

Proof. (i) follows easily from (2.2.2) and (ii) follows easily from (i). \square

Corollary 2.2.10. *Suppose that $\tilde{\mathfrak{m}}$ is a flat V -module. Then we have :*

i) the functor $M \mapsto M_{\dagger}$ is exact;

ii) the localisation functor $R\text{-Mod} \rightarrow R^a\text{-Mod}$ sends injectives to injectives.

Proof. By proposition 2.2.9 it follows that $M \mapsto M_{\dagger}$ is right exact. To show that it is also left exact when $\tilde{\mathfrak{m}}$ is a flat V -module, it suffices to remark that $M \mapsto M_*$ is left exact. Now, by (i), the functor $M \mapsto M^a$ is right adjoint to an exact functor, so (ii) is clear. \square

Next, let B be any A -algebra. The multiplication on B_* is inherited by B_{\dagger} , which is therefore a non-unital ring in a natural way. We endow the V -module $V \oplus B_{\dagger}$ with the ring structure determined by the rule: $(v, b) \cdot (v', b') = (v \cdot v', v \cdot b' + v' \cdot b + b \cdot b')$ for all $v, v' \in V$ and $b, b' \in B_{\dagger}$. Then $V \oplus B_{\dagger}$ is a (unital) ring. Let $\mu_{\mathfrak{m}} : \tilde{\mathfrak{m}} \rightarrow \mathfrak{m}$ be the map defined by $x \otimes y \mapsto xy$ for all $x, y \in \mathfrak{m}$; we notice that the subset of all elements of the form $(\mu(s), -s \otimes \underline{1})$ (for arbitrary $s \in \tilde{\mathfrak{m}}$) forms an ideal I of $V \oplus B_{\dagger}$. Set $B_{\dagger\dagger} = (V \oplus B_{\dagger})/I$. Thus we have a sequence of V -modules

$$(2.2.11) \quad 0 \rightarrow \tilde{\mathfrak{m}} \rightarrow V \oplus B_{\dagger} \rightarrow B_{\dagger\dagger} \rightarrow 0$$

which in general is only right exact.

Definition 2.2.12. We say that B is an *exact A -algebra* if the sequence (2.2.11) is exact.

Remark 2.2.13. Notice that if $\tilde{\mathfrak{m}} \xrightarrow{\sim} \mathfrak{m}$ (e.g. when \mathfrak{m} is flat), then all A -algebras are exact. In the general case, if B is any A -algebra, then $V^a \times B$ is always exact. Indeed, we have $(V^a \times B)_* \simeq V_*^a \times B_*$ and, by remark 2.1.3(i), $\tilde{\mathfrak{m}} \otimes_V V_*^a \simeq \tilde{\mathfrak{m}}$.

Clearly we have a natural isomorphism $B \simeq B_{||}^a$.

Proposition 2.2.14. *The functor $B \mapsto B_{||}$ is left adjoint to the localisation functor $A_{||}\text{-Alg} \rightarrow A\text{-Alg}$.*

Proof. Let B be an A -algebra, C an $A_{||}$ -algebra and $\phi : B \rightarrow C^a$ a morphism of A -algebras. By proposition 2.2.9 we obtain a natural A_* -linear morphism $B_{||} \rightarrow C$. Together with the structure morphism $V \rightarrow C$ this yields a map $\tilde{\phi} : V \oplus B_{||} \rightarrow C$ which is easily seen to be a ring homomorphism. It is equally clear that the ideal I defined above is mapped to zero by $\tilde{\phi}$, hence the latter factors through a map of $A_{||}$ -algebras $B_{||} \rightarrow C$. Conversely, such a map induces a morphism of A -algebras $B \rightarrow C^a$ just by taking localisation. It is easy to check that the two procedures are inverse to each other, which shows the assertion. \square

Remark 2.2.15. The functor of almost elements commutes with arbitrary limits, because all right adjoints do. It does not in general commute with colimits, not even with arbitrary infinite direct sums. Dually, the functors $M \mapsto M_{||}$ and $B \mapsto B_{||}$ commute with all colimits. In particular, the latter commutes with tensor products.

2.3. Almost homological algebra. In this section we fix an almost V -algebra A and we consider various constructions in the category of A -modules.

Remark 2.3.1. i) Let M_1, M_2 be two A -modules. By proposition 2.2.4 it is clear that a morphism $\phi : M_1 \rightarrow M_2$ of A -modules is uniquely determined by the induced morphism $M_{1*} \rightarrow M_{2*}$.

ii) It is a bit tricky to deal with preimages of almost elements under morphisms: for instance, if $\phi : M_1 \rightarrow M_2$ is an epimorphism (by which we mean that $\text{Coker}(\phi) \simeq 0$) and $m_2 \in M_{2*}$, then it is not true in general that we can find an almost element $m_1 \in M_{1*}$ such that $\phi_*(m_1) = m_2$. What remains true is that for arbitrary $\varepsilon \in \mathfrak{m}$ we can find m_1 such that $\phi_*(m_1) = \varepsilon \cdot m_2$.

The abelian category $A\text{-Mod}$ satisfies axiom (AB5) (see e.g. [22] (§A.4)) and it has a generator, namely the object A itself. It then follows by a general result that $A\text{-Mod}$ has enough injectives. By corollary 2.2.7 any inverse system $\{M_n \mid n \in \mathbb{N}\}$ of A -modules has an (inverse) limit $\lim_{n \in \mathbb{N}} M_n$. As usual, we denote by \lim^1 the right derived functor of the inverse limit functor. Notice that [22] (Cor. 3.5.4) holds in the almost case since axiom (AB4*) holds in $A\text{-Mod}$ (on the other hand, it is not clear whether [22] (Lemma 3.5.3) holds under (AB4*), since the proof uses elements).

Lemma 2.3.2. *Let $\{M_n ; \phi_n : M_n \rightarrow M_{n+1} \mid n \in \mathbb{N}\}$ (resp. $\{N_n ; \psi_n : N_{n+1} \rightarrow N_n \mid n \in \mathbb{N}\}$) be a direct (resp. inverse) system of A -modules and morphisms and $\{\varepsilon_n \mid n \in \mathbb{N}\}$ a sequence of ideals of V converging to V (for the uniform structure introduced in section 2.1).*

i) *If $\varepsilon_n \cdot M_n = 0$ for all $n \in \mathbb{N}$ then $\text{colim}_{n \in \mathbb{N}} M_n \simeq 0$.*

ii) *If $\varepsilon_n \cdot N_n = 0$ for all $n \in \mathbb{N}$ then $\lim_{n \in \mathbb{N}} N_n \simeq 0 \simeq \lim_{n \in \mathbb{N}}^1 N_n$.*

iii) *If $\varepsilon_n \cdot \text{Coker}(\psi_n) = 0$ for all $n \in \mathbb{N}$ and $\prod_{j=0}^{\infty} \varepsilon_j$ is a Cauchy product, then $\lim_{n \in \mathbb{N}}^1 N_n \simeq 0$.*

Proof. (i) and (ii) : we remark only that $\lim_{n \in \mathbb{N}}^1 N_n \simeq \lim_{n \in \mathbb{N}}^1 N_{n+p}$ for all $p \in \mathbb{N}$ and leave the details to the reader. We prove (iii). From [22] (Cor. 3.5.4) it follows easily that $(\lim_{n \in \mathbb{N}}^1 N_{n*})^a \simeq \lim_{n \in \mathbb{N}}^1 N_n$. It then suffices to show that $\lim_{n \in \mathbb{N}}^1 N_{n*}$ is almost zero. We have $\varepsilon_n^2 \cdot \text{Coker}(\psi_{n*}) = 0$ and the product $\prod_{j=0}^{\infty} (\varepsilon_j^2)$ is again a Cauchy product. Next let $N'_n = \bigcap_{p \geq 0} \text{Im}(N_{n+p*} \rightarrow N_{n*})$. If $J_n = \bigcap_{p \geq 0} (\varepsilon_n \cdot \varepsilon_{n+1} \cdot \dots \cdot \varepsilon_{n+p})^2$ then $J_n \cdot N_{n*} \subset N'_n$ and $\lim_{n \rightarrow \infty} J_n = V$. In view of (ii), $\lim_{n \in \mathbb{N}}^1 N_{n*}/N'_n$ is almost zero, hence we reduce to showing that $\lim_{n \in \mathbb{N}}^1 N'_n$ is almost zero. But

$$J_{n+p+q} \cdot N'_n \subset \text{Im}(N'_{n+p+q} \rightarrow N'_n) \subset \text{Im}(N'_{n+p} \rightarrow N'_n)$$

for all $n, p, q \in \mathbb{N}$. On the other hand, by remark 2.1.2 we get $\bigcup_{q=0}^{\infty} \mathfrak{m} \cdot J_{n+p+q} = \mathfrak{m}$, hence $\mathfrak{m} \cdot N'_n \subset \text{Im}(N'_{n+p} \rightarrow N'_n)$ and finally $\mathfrak{m} \cdot N'_n = \mathfrak{m}^2 \cdot N'_n \subset \text{Im}(\mathfrak{m} \cdot N'_{n+p} \rightarrow \mathfrak{m} \cdot N'_n)$ which means that $\{\mathfrak{m} \cdot N'_n\}$ is a surjective inverse system, so its \lim^1 vanishes and the result follows. \square

Example 2.3.3. Let (V, \mathfrak{m}) be as in example 2.1.1. Then every ideal in V is principal, so in the situation of the lemma we can write $\varepsilon_j = (x_j)$ for some $x_j \in V$. Then the hypothesis in (iii) can be stated by saying that there exists $c \in \mathbb{N}$ such that $x_j \neq 0$ for all $j \geq c$ and the sequence $n \mapsto \sum_{j=c}^n \nu(x_j)$ is Cauchy in Γ .

Definition 2.3.4. Let M be an A -module.

i) We say that M is *flat* (resp. *faithfully flat*) if the functor $N \mapsto M \otimes_A N$, from the category of A -modules to itself is exact (resp. exact and faithful). M is *almost projective* if the functor $N \mapsto \text{alHom}_A(M, N)$ is exact.

ii) We say that M is *finitely generated* if there exists a positive integer n and an epimorphism $A^n \rightarrow M$. We say that M is *almost finitely generated* if, for arbitrary $\varepsilon \in \mathfrak{m}$, there exists a finitely generated submodule $M_\varepsilon \subset M$ such that $\varepsilon \cdot M \subset M_\varepsilon$.

iii) We say that M is *almost finitely presented* if, for arbitrary $\varepsilon, \delta \in \mathfrak{m}$ there exist positive integers $n = n(\varepsilon), m = m(\varepsilon)$ and a three term complex $A^m \xrightarrow{\psi_\varepsilon} A^n \xrightarrow{\phi_\varepsilon} M$ with $\varepsilon \cdot \text{Coker}(\phi_\varepsilon) = 0$ and $\delta \cdot \text{Ker}(\phi_\varepsilon) \subset \text{Im}(\psi_\varepsilon)$.

Proposition 2.3.5. (i) An A -module M is almost finitely generated if and only if for every finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ there exists a finitely generated submodule $M_0 \subset M$ such that $\mathfrak{m}_0 \cdot M \subset M_0$.

(ii) An A -module is almost finitely presented if and only if, for every finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ there is a complex $A^m \xrightarrow{\psi} A^n \xrightarrow{\phi} M$ with $\mathfrak{m}_0 \cdot \text{Coker}(\phi) = 0$ and $\mathfrak{m}_0 \cdot \text{Ker}(\phi) \subset \text{Im}(\psi)$.

Proof. (i) is easy and we leave it to the reader. To prove (ii), take a finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m}$ such that $\mathfrak{m}_0 \subset \mathfrak{m} \cdot \mathfrak{m}_1$, pick a morphism $\phi : A^n \rightarrow M$ whose cokernel is annihilated by \mathfrak{m}_1 , and apply the following lemma 2.3.6. \square

Lemma 2.3.6. If M is almost finitely presented and $\phi : F \rightarrow M$ is a morphism with $F \simeq A^n$, then for every finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m} \cdot \text{Ann}_V(\text{Coker}(\phi))$ there is a finitely generated submodule of $\text{Ker}(\phi)$ containing $\mathfrak{m}_1 \cdot \text{Ker}(\phi)$.

Proof. We need the following

Claim 2.3.7. Let F_1 be a finitely generated A -module and suppose that we are given $a, b \in V$ and a (not necessarily commutative) diagram

$$\begin{array}{ccc} F_1 & \xrightarrow{p} & M \\ \psi \downarrow \phi & \nearrow q & \\ F_2 & & \end{array}$$

such that $q \circ \phi = a \cdot p$, $p \circ \psi = b \cdot q$. Let $I \subset V$ be an ideal such that $\text{Ker}(q)$ has a finitely generated submodule containing $I \cdot \text{Ker}(q)$. Then $\text{Ker}(p)$ has a finitely generated submodule containing $a \cdot b \cdot I \cdot \text{Ker}(p)$.

Proof of the claim: Let R be the submodule of $\text{Ker}(q)$ given by the assumption. We have $\text{Im}(\psi \circ \phi - a \cdot b \cdot \mathbf{1}_{F_1}) \subset \text{Ker}(p)$ and $\psi(R) \subset \text{Ker}(p)$. We take $R_1 = \text{Im}(\psi \circ \phi - a \cdot b \cdot \mathbf{1}_{F_1}) + \psi(R)$. Clearly $\phi(\text{Ker}(p)) \subset \text{Ker}(q)$, so $I \cdot \phi(\text{Ker}(p)) \subset R$, hence $I \cdot \psi \circ \phi(\text{Ker}(p)) \subset \psi(R)$ and finally $a \cdot b \cdot I \cdot \text{Ker}(p) \subset R_1$.

Now, let $\delta \in \text{Ann}_V(\text{Coker}(\phi))$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \mathfrak{m}$. By assumption there is a complex $A^r \xrightarrow{t} A^s \xrightarrow{q} M$ with $\varepsilon_1 \cdot \text{Coker}(q) = 0$, $\varepsilon_2 \cdot \text{Ker}(q) \subset \text{Im}(t)$. Letting $F_1 = F$, $F_2 = A^s$,

$a = \varepsilon_1 \cdot \varepsilon_3$, $b = \varepsilon_4 \cdot \delta$, one checks easily that ψ and ϕ can be given such that all the assumptions of the above claim are fulfilled. So, with $I = \varepsilon_2 \cdot V$ we get that $\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 \cdot \varepsilon_4 \cdot \delta \cdot \text{Ker}(\phi)$ lies in a finitely generated submodule of $\text{Ker}(\phi)$. But \mathfrak{m}_1 is contained in an ideal generated by finitely many such products $\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 \cdot \varepsilon_4 \cdot \delta$. \square

The following proposition generalises a well-known characterization of finitely presented modules over usual rings.

Proposition 2.3.8. *Let M be an A -module.*

i) *M is almost finitely generated if and only if, for every filtered inductive system $(N_\lambda, \phi_{\lambda\mu})$ (indexed by a directed set Λ) the natural morphism*

$$\nu : \text{colim}_{\Lambda} \text{alHom}_A(M, N_\lambda) \rightarrow \text{alHom}_A(M, \text{colim}_{\Lambda} N_\lambda)$$

is a monomorphism.

ii) *M is almost finitely presented if and only if for every filtered inductive system as above, ν is an isomorphism.*

Proof. The “only if” part in (i) (resp. (ii)) is first checked when M is finitely generated (resp. finitely presented) and then extended to the general case. We leave the details to the reader and we proceed to verify the “if” part. For (i), choose a set I and an epimorphism $p : A^{(I)} \rightarrow M$. Let Λ be the directed set of finite subsets of I , ordered by inclusion. For $S \in \Lambda$, let $M_S = p(A^S)$. Then $\text{colim}_{\Lambda} (M/M_S) = 0$, so the assumption gives $\text{colim}_{\Lambda} \text{alHom}_A(M, M/M_S) = 0$, i.e. $\text{colim}_{\Lambda} \text{Hom}_A(M, M/M_S) = 0$ is almost zero, so, for every $\varepsilon \in \mathfrak{m}$, the image of $\varepsilon \cdot \mathbf{1}_M$ in the above colimit is 0, i.e. there exists $S \in \Lambda$ such that $\varepsilon \cdot M \subset M_S$, which proves the contention. For (ii), we present M as a filtered colimit $\text{colim}_{\Lambda} M_\lambda$, where each M_λ is finitely presented (this can be done e.g. by taking such a presentation of the A_* -module M_* and applying $N \mapsto N^a$). The assumption of (ii) gives that $\text{colim}_{\Lambda} \text{Hom}_A(M, M_\lambda) \rightarrow \text{Hom}_A(M, M)$ is an almost isomorphism, hence, for every $\varepsilon \in \mathfrak{m}$ there is $\lambda \in \Lambda$ and $\phi_\varepsilon : M \rightarrow M_\lambda$ such that $p_\lambda \circ \phi_\varepsilon = \varepsilon \cdot \mathbf{1}_M$, where $p_\lambda : M_\lambda \rightarrow M$ is the natural morphism to the colimit. If such a ϕ_ε exists for λ , then it exists for every $\mu \geq \lambda$. Hence, if $\mathfrak{m}_0 \subset \mathfrak{m}$ is a finitely generated subideal, say $\mathfrak{m}_0 = \sum_j^k V\varepsilon_j$, then there exist $\lambda \in \Lambda$ and $\phi_i : M \rightarrow M_\lambda$ such that $p_\lambda \circ \phi_i = \varepsilon_i \cdot \mathbf{1}_M$ for $i = 1, \dots, k$. Hence $\text{Im}(\phi_i \circ p_\lambda - \varepsilon_i \cdot \mathbf{1}_{M_\lambda})$ is contained in $\text{Ker}(p_\lambda)$ and contains $\varepsilon_i \cdot \text{Ker}(p_\lambda)$. Hence $\text{Ker}(p_\lambda)$ has a finitely generated submodule L containing $\mathfrak{m}_0 \cdot \text{Ker}(p_\lambda)$. Choose a presentation $A^m \rightarrow A^n \xrightarrow{\pi} M_\lambda$. Then one can lift $\mathfrak{m}_0 \cdot L$ to a finitely generated submodule L' of A^n . Then $\text{Ker}(\pi) + L'$ is a finitely generated submodule of $\text{Ker}(p_\lambda \circ \pi)$ containing $\mathfrak{m}_0^2 \cdot \text{Ker}(p_\lambda \circ \pi)$. Since we also have $\mathfrak{m}_0 \cdot \text{Coker}(p_\lambda \circ \pi) = 0$ and \mathfrak{m}_0 is arbitrary, the conclusion follows from proposition 2.3.5. \square

Lemma 2.3.9. *Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of A -modules. Then:*

i) *If M' , M'' are almost finitely generated (resp. presented) then so is M .*

ii) *If M is almost finitely presented, then M'' is almost finitely presented if and only if M' is almost finitely generated.*

Proof. These facts can be deduced from proposition 2.3.8 and remark 2.3.11(iii), or proved directly. \square

Lemma 2.3.10. *Let \mathbf{P} be one of the properties : “flat”, “almost projective”, “almost finitely generated”, “almost finitely presented”. If B is a \mathbf{P} A -algebra, and M is a \mathbf{P} B -module, then M is \mathbf{P} as an A -module.*

Proof. Left to the reader. \square

Let R be a V -algebra and M a flat (resp. faithfully flat) R -module (in the usual sense, see [18] p.45). Then M^a is a flat (resp. faithfully flat) R^a -module. Indeed, the functor $M \otimes_R -$ preserves the Serre subcategory of almost zero modules, so by general facts it induces an exact functor on the localized categories (cp. [8] p.369). For the faithfulness we have to show that an R -module N is almost zero whenever $M \otimes_R N$ is almost zero. However, $M \otimes_R N$ is almost zero $\Leftrightarrow M \otimes_R (\mathfrak{m} \otimes_V N) = 0 \Leftrightarrow \mathfrak{m} \otimes_V N = 0 \Leftrightarrow N$ is almost zero. It is clear that $A\text{-Mod}$ has enough almost projective (resp. flat) objects. Let R be a V -algebra. The localisation functor induces a functor $G : \mathbf{D}(R) \rightarrow \mathbf{D}(R^a)$ and, in view of corollary 2.2.10, $M \mapsto M_!$ induces a functor $F : \mathbf{D}(R^a) \rightarrow \mathbf{D}(R)$. We have a natural isomorphism $G \circ F \simeq \mathbf{1}_{\mathbf{D}(R^a)}$ and a natural transformation $F \circ G \rightarrow \mathbf{1}_{\mathbf{D}(R)}$. These satisfy the triangular identities of [17] (p.83) so F is a left adjoint to G . If Σ denotes the multiplicative set of morphisms in $\mathbf{D}(R)$ which induce almost isomorphisms on the cohomology modules, then the localised category $\Sigma^{-1}\mathbf{D}(R)$ exists (see e.g. [22] (Th.10.3.7)) and by the same argument we get an equivalence of categories $\Sigma^{-1}\mathbf{D}(R) \simeq \mathbf{D}(R^a)$.

Given an A -module M , we can derive the functors $M \otimes_A -$ (resp. $\text{alHom}_A(M, -)$, resp. $\text{alHom}_A(-, M)$) by taking flat (resp. injective, resp. almost projective) resolutions : one remarks that bounded above exact complexes of flat (resp. almost projective) A -modules are acyclic for the functor $M \otimes_A -$ (resp. $\text{alHom}_A(-, M)$) (recall the standard argument: if F_\bullet is a complex of flat A -modules, let Φ_\bullet be a flat resolution of M ; then $\text{Tot}(\Phi_\bullet \otimes_A F_\bullet) \rightarrow M \otimes_A F_\bullet$ is a quasi-isomorphism since it is so on rows, and $\text{Tot}(\Phi_\bullet \otimes_A F_\bullet)$ is acyclic since its columns are; similarly, if P_\bullet is a complex of almost projective objects, one considers the double complex $\text{alHom}_A(P_\bullet, J^\bullet)$ where J^\bullet is an injective resolution of M ; cp. [22] §2.7); then one uses the construction detailed in [22] (Th.10.5.9). We denote by $\text{Tor}_i^A(M, -)$ (resp. $\text{alExt}_A^i(M, -)$, resp. $\text{alExt}_A^i(-, M)$) the corresponding derived functors. If $A = R^a$ for some V -algebra R , we obtain easily natural isomorphisms $\text{Tor}_i^R(M, N)^a \simeq \text{Tor}_i^A(M^a, N^a)$ for all R -modules M, N . A similar result holds for $\text{Ext}_R^i(M, N)$.

Remark 2.3.11. i) Clearly, an A -module M is flat (resp. almost projective) if and only if $\text{Tor}_i^A(M, N) = 0$ (resp. $\text{alExt}_A^i(M, N) = 0$) for all A -modules N and all $i > 0$.

ii) Let M, N be two flat (resp. almost projective) A -modules. Then $M \otimes_A N$ is a flat (resp. almost projective) A -module and for any A -algebra B , the B -module $B \otimes_A M$ is flat (resp. almost projective).

iii) Resume the notation of proposition 2.3.8. If M is almost finitely presented, then one has also that the natural morphism $\text{colim}_\Lambda \text{alExt}_A^1(M, N_\lambda) \rightarrow \text{alExt}_A^1(M, \text{colim}_\Lambda N_\lambda)$ is a monomorphism. This is deduced from proposition 2.3.8(ii), using the fact that (N_λ) can be injected into an inductive system (J_λ) of injective almost modules (e.g. $J_\lambda = E^{\text{Hom}_A(N_\lambda, E)}$, where E is an injective cogenerator for $A\text{-Mod}$), and by applying alExt sequences.

Lemma 2.3.12. *Let M be an almost finitely generated A -module. Consider the following properties:*

i) M is almost projective.

ii) For arbitrary $\varepsilon \in \mathfrak{m}$ there exist $n(\varepsilon) \in \mathbb{N}$ and A -linear morphisms

$$(2.3.13) \quad M \xrightarrow{u_\varepsilon} A^{n(\varepsilon)} \xrightarrow{v_\varepsilon} M$$

such that $v_\varepsilon \circ u_\varepsilon = \varepsilon \cdot \mathbf{1}_M$.

iii) M is flat.

Then (i) \Leftrightarrow (ii) \Rightarrow (iii).

Proof. (ii) \Rightarrow (i): for given $\varepsilon \in \mathfrak{m}$, we consider any A -module N and we apply the functor $\text{alExt}_A^i(-, N)$ to (2.3.13) :

$$\text{alExt}_A^i(M, N) \longrightarrow \text{alExt}_A^i(A^{n(\varepsilon)}, N) \simeq 0 \longrightarrow \text{alExt}_A^i(M, N)$$

which implies $\varepsilon \cdot \text{alExt}^i(M, N) = 0$ for all $i > 0$. Since ε is arbitrary, (i) follows from remark 2.3.11(i).

(i) \Rightarrow (ii): by hypothesis, for arbitrary $\varepsilon \in \mathfrak{m}$ we can find $n = n(\varepsilon)$ and a morphism $\phi_\varepsilon : A^n \rightarrow M$ such that $\varepsilon \cdot \text{Coker}(\phi_\varepsilon) = 0$. Let M_ε be the image of ϕ_ε , so that ϕ_ε factors as $A^{n(\varepsilon)} \xrightarrow{\psi_\varepsilon} M_\varepsilon \xrightarrow{j_\varepsilon} M$. Also $\varepsilon \cdot \mathbf{1}_M : M \rightarrow M$ factors as $M \xrightarrow{\gamma_\varepsilon} M_\varepsilon \xrightarrow{j_\varepsilon} M$. Since by hypothesis M is almost projective, the natural morphism induced by ψ_ε

$$\text{alHom}_A(M, A^n) \xrightarrow{\psi_\varepsilon^*} \text{alHom}_A(M, M_\varepsilon)$$

is an epimorphism. Then for arbitrary $\delta \in \mathfrak{m}$ the morphism $\delta \cdot \gamma_\varepsilon$ is in the image of ψ_ε^* , in other words, there exists an A -linear morphism $u_{\varepsilon\delta} : M \rightarrow A^n$ such that $\psi_\varepsilon \circ u_{\varepsilon\delta} = \delta \cdot \gamma_\varepsilon$. If now we take $v_{\varepsilon\delta} = \phi_\varepsilon$, it is clear that $v_{\varepsilon\delta} \circ u_{\varepsilon\delta} = \varepsilon \cdot \delta \cdot \mathbf{1}_M$. This proves (ii), since the $\varepsilon \in \mathfrak{m}$ satisfying the assertion of (ii) form an ideal.

(ii) \Rightarrow (iii): for a given A -module N , apply the functor $\text{Tor}_i^A(-, N)$ to the sequence (2.3.13). This yields $\varepsilon \cdot \text{Tor}_i^A(M, N) = 0$. Now the claim follows from remark 2.3.11(i). \square

There is a converse to lemma 2.3.12 in case M is almost finitely presented. Before stating it, we need the following lemma.

Lemma 2.3.14. *Let R be any ring, M any R -module and $C = \text{Coker}(\phi : R^n \rightarrow R^m)$ any finitely presented (left) R -module. Let $C' = \text{Coker}(\phi^* : R^m \rightarrow R^n)$ be the cokernel of the transpose of the map ϕ . Then there is a natural isomorphism*

$$\text{Tor}_1^R(C', M) \simeq \text{Hom}_R(C, M) / \text{Im}(\text{Hom}_R(C, R) \otimes_R M).$$

Proof. We have a spectral sequence :

$$E_{ij}^2 = \text{Tor}_i^R(H_j(\text{Cone}(\phi^*)), M) \Rightarrow H_{i+j}(\text{Cone}(\phi^*) \otimes_R M).$$

On the other hand we have also natural isomorphisms

$$\text{Cone}(\phi^*) \otimes_R M \simeq \text{Hom}_R(\text{Cone}(\phi), R)[1] \otimes_R M \simeq \text{Hom}_R(\text{Cone}(\phi), M)[1].$$

Hence :

$$\begin{aligned} E_{10}^2 &\simeq E_{10}^\infty \simeq H_1(\text{Cone}(\phi^*) \otimes_R M) / E_{01}^\infty \simeq H^0(\text{Hom}_R(\text{Cone}(\phi), M)) / \text{Im}(E_{01}^2) \\ &\simeq \text{Hom}_R(C, M) / \text{Im}(\text{Hom}_R(C, R) \otimes_R M) \end{aligned}$$

which is the claim. \square

Proposition 2.3.15. (i) *Every almost finitely generated almost projective A -module is almost finitely presented.*

(ii) *Every almost finitely presented flat A -module is almost projective.*

Proof. (ii) : let M be such an A -module. Let $\varepsilon, \delta \in \mathfrak{m}$ and pick a three term complex

$$A^m \xrightarrow{\psi} A^n \xrightarrow{\phi} M$$

such that $\varepsilon \cdot \text{Coker}(\phi) = \delta \cdot \text{Ker}(\phi) / \text{Im}(\psi) = 0$. Set $P = \text{Coker}(\psi_*)$; this is a finitely presented A_* -module and ϕ_* factors through a morphism $\bar{\phi}_* : P \rightarrow M_*$. Let $\gamma \in \mathfrak{m}$; from lemma 2.3.14 we see that $\gamma \cdot \bar{\phi}_*$ is the image of some element $\sum_{j=1}^n \phi_j \otimes m_j \in \text{Hom}_{A_*}(P, A_*) \otimes_{A_*} M_*$. If we define $L = A_*^n$ and $v : P \rightarrow L$, $w : L \rightarrow M_*$ by $v(x) = (\phi_1(x), \dots, \phi_n(x))$ and $w(y_1, \dots, y_n) = \sum_{j=1}^n y_j \cdot m_j$, then clearly $\gamma \cdot \bar{\phi}_* = w \circ v$. Let $K = \text{Ker}(\bar{\phi}_*)$. Then $\delta \cdot K^a = 0$ and the map $\delta \cdot \mathbf{1}_{P^a}$ factors through a morphism $\sigma : (P/K)^a \rightarrow P^a$. Similarly the map $\varepsilon \cdot \mathbf{1}_M$ factors through a morphism $\lambda : M \rightarrow (P/K)^a$. Let $\alpha = v^a \circ \sigma \circ \lambda : M \rightarrow L^a$ and $\beta = w^a : L^a \rightarrow M$. The reader can check that $\beta \circ \alpha = \varepsilon \cdot \delta \cdot \gamma \cdot \mathbf{1}_M$. By lemma 2.3.12 the claim follows.

(i) : let P be such an almost finitely generated almost projective A -module. For any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ pick a morphism $\phi : A^r \rightarrow P$ such that $\mathfrak{m}_0 \cdot \text{Coker}(\phi) = 0$. If

$\varepsilon_1, \dots, \varepsilon_k$ is a set of generators for \mathfrak{m}_0 , a standard argument shows that, for any $i \leq k$, $\varepsilon_i \cdot \mathbf{1}_P$ lifts to a morphism $\psi_i : P \rightarrow A^r / \text{Ker}(\phi)$; then, since P is almost projective, $\varepsilon_j \psi_i$ lifts to a morphism $\psi_{ij} : P \rightarrow A^r$. Now claim 2.3.7 applies with $F_1 = A^r$, $F_2 = M = P$, $p = \phi$, $q = \mathbf{1}_P$ and $\psi = \psi_{ij}$ and shows that $\text{Ker}(\phi)$ has a finitely generated submodule M_{ij} containing $\varepsilon_i \cdot \varepsilon_j \cdot \text{Ker}(\phi)$. Then the span of all such M_{ij} is a finitely generated submodule of $\text{Ker}(\phi)$ containing $\mathfrak{m}_0^2 \cdot \text{Ker}(\phi)$. By proposition 2.3.5(ii), the claim follows. \square

Definition 2.3.16. For an A -module M , the *dual A -module* of M is the A -module $M^* = \text{alHom}_A(M, A)$. M is *reflexive* if the natural morphism

$$(2.3.17) \quad M \rightarrow (M^*)^* \quad m \mapsto (f \mapsto f(m))$$

is an isomorphism of A -modules.

Remark 2.3.18. Notice that if B is an A -algebra and M any B -module, then by “restriction of scalars” M is also an A -module and the dual A -module M^* has a natural structure of B -module. This is defined by the rule $(b \cdot f)(m) = f(b \cdot m)$ ($b \in B_*$, $m \in M_*$ and $f \in M^*$). With respect to this structure (2.3.17) becomes a B -linear morphism. Incidentally, notice that the two meanings of “ M^* ” coincide, i.e. $(M_*)^* \simeq (M^*)^*$.

Proposition 2.3.19. Let P be an almost projective A -module and denote by I_P the image of the natural “evaluation morphism” $P \otimes_A P^* \rightarrow A$.

- i) For every morphism of algebras $A \rightarrow B$ we have $I_{B \otimes_A P} = I_P \cdot B$.
- ii) $I_P = I_P^2$.
- iii) $P = 0$ if and only if $I_P = 0$.
- iv) P is faithfully flat if and only if $I_P = A$.

Proof. Pick an indexing set I large enough, and an epimorphism $\phi : F = A^{(I)} \rightarrow P$. For every $i \in I$ we have the standard morphisms $A \xrightarrow{e_i} F \xrightarrow{\pi_i} A$ such that $\pi_i \circ e_j = \delta_{ij} \cdot \mathbf{1}_A$ and $\sum_{i \in I} e_i \circ \pi_i = \mathbf{1}_F$. For every $x \in \mathfrak{m}$ choose $\psi_x \in \text{Hom}_A(P, F)$ such that $\phi \circ \psi_x = x \cdot \mathbf{1}_P$. It is easy to check that I_P is generated by the almost elements $\pi_i \circ \psi_x \circ \phi \circ e_j$ ($i, j \in I$, $x \in \mathfrak{m}$). (i) follows already. For (iii), the “only if” is clear; if $I_P = 0$, then $\psi_x \circ \phi = 0$ for all $x \in \mathfrak{m}$, hence $\psi_x = 0$ and therefore $x \cdot \mathbf{1}_P = 0$. Next, notice that, from (i) and (iii) we derive $P/(I_P \cdot P) = 0$, i.e. $P = I_P \cdot P$, so (ii) follows directly from the definition of I_P . Since P is flat, to show (iv) we have only to verify that the functor $M \mapsto P \otimes_A M$ is faithful. To this purpose, it suffices to check that $P \otimes_A (A/J) \neq 0$ for every proper ideal J of A . This follows easily from (i) and (iii). \square

If E, F and N are A -modules, there is a natural morphism :

$$(2.3.20) \quad E \otimes_A \text{alHom}_A(F, N) \rightarrow \text{alHom}_A(F, E \otimes_A N).$$

Lemma 2.3.21. (i) The morphism (2.3.20) is an isomorphism in the following cases :

- a) when E is flat and F is almost finitely presented;
- b) when either E or F is almost finitely generated and almost projective;
- c) when F is almost projective and E is almost finitely presented;
- d) when E is almost projective and F is almost finitely generated.

(ii) The morphism (2.3.20) is a monomorphism in the following cases :

- a) when E is flat and F is almost finitely generated;
- b) when E is almost projective.

(iii) The morphism (2.3.20) is an epimorphism when F is almost projective and E is almost finitely generated.

Proof. If $F \simeq A^{(I)}$ for some finite set I , then $\text{alHom}_A(F, N) \simeq N^{(I)}$ and the claims are obvious. More generally, if F is almost projective and almost finitely generated, for any $\varepsilon \in \mathfrak{m}$ there

exists a finite set $I = I(\varepsilon)$ and morphisms

$$(2.3.22) \quad F \xrightarrow{u_\varepsilon} A^{(I)} \xrightarrow{v_\varepsilon} F$$

such that $v_\varepsilon \circ u_\varepsilon = \varepsilon \cdot \mathbf{1}_F$. We apply the natural transformation

$$E \otimes_A \text{alHom}_A(-, N) \rightarrow \text{alHom}_A(-, E \otimes_A N)$$

to (2.3.22) : an easy diagram chase allows then to conclude that the kernel and cokernel of (2.3.20) are killed by ε . As ε is arbitrary, it follows that (2.3.20) is an isomorphism in this case. An analogous argument works when E is almost finitely generated almost projective, so we get (i.b). If F is only almost projective, then we still have morphisms of the type (2.3.22), but now $I(\varepsilon)$ is no longer necessarily finite. However, the cokernels of the induced morphisms $\mathbf{1}_E \otimes u_\varepsilon$ and $\text{alHom}_A(v_\varepsilon, E \otimes_A N)$ are still annihilated by ε . Hence, to show (iii) (resp. (i.c)) it suffices to consider the case when F is free and E is almost finitely generated (resp. presented). By passing to almost elements, we can further reduce to the analogous question for usual rings and modules, and by the usual juggling we can even replace E by a finitely generated (resp. presented) A_* -module and F by a free A_* -module. This case is easily dealt with, and (iii) and (i.c) follow. Case (i.d) (resp. (ii.b)) is similar : one considers almost elements and replaces E_* by a free A_* -module (resp. and F_* by a finitely generated A_* -module). In case (ii.a) (resp. (i.a)), for every finitely generated submodule \mathfrak{m}_0 of \mathfrak{m} we can find, by proposition 2.3.5, a finitely generated (resp. presented) A -module F_0 and a morphism $F_0 \rightarrow F$ whose kernel and cokernel are annihilated by \mathfrak{m}_0 . It follows easily that we can replace F by F_0 and suppose that F is finitely generated (resp. presented). Then the argument in [2] (Ch.I §2 Prop.10) can be taken over *verbatim* to show (ii.a) (resp. (i.a)). \square

Lemma 2.3.23. *i) Let P be an A -module and B an A -algebra. If either P or B is almost finitely generated almost projective as an A -module, then the natural morphism*

$$(2.3.24) \quad B \otimes_A \text{alHom}_A(P, N) \rightarrow \text{alHom}_B(B \otimes_A P, B \otimes_A N)$$

is an isomorphism for all A -modules N .

ii) Every almost projective almost finitely generated A -module is reflexive.

Proof. (i) is an easy consequence of lemma 2.3.21(i.b). To prove (ii), we we apply the natural transformation (2.3.17) to (2.3.22) : by diagram chase one sees that the kernel and cokernel of the morphism $F \rightarrow (F^*)^*$ are killed by ε . \square

Lemma 2.3.25. *Let $\{M_n ; \phi_n : M_n \rightarrow M_{n+1} \mid n \in \mathbb{N}\}$ be a direct system of A -modules and suppose there exist sequences $\{\varepsilon_n \mid n \in \mathbb{N}\}$ and $\{\delta_n \mid n \in \mathbb{N}\}$ of ideals of V such that*

i) $\lim_{n \rightarrow \infty} \varepsilon_n = V$ (convergence for the uniform structure on ideals of V) and $\prod_{j=0}^{\infty} \delta_j$ is a Cauchy product;

ii) for all $n \in \mathbb{N}$ there exist integers $N(n)$ and morphisms of A -modules $\psi_n : A^{N(n)} \rightarrow M_n$ such that $\varepsilon_n \cdot \text{Coker}(\psi_n) = 0$;

iii) $\delta_n \cdot \text{Coker}(\phi_n) = 0$ for all $n \in \mathbb{N}$.

Then $\text{colim}_{n \in \mathbb{N}} M_n$ is an almost finitely generated A -module.

Proof. Let $M = \text{colim}_{n \in \mathbb{N}} M_n$. For any $n \in \mathbb{N}$ let $a_n = \bigcap_{m \geq 0} (\prod_{j=n}^{n+m} \delta_j)$. Then $\lim_{n \rightarrow \infty} a_n = V$. For $m > n$ set $\phi_{n,m} = \phi_m \circ \dots \circ \phi_{n+1} \circ \phi_n : M_n \rightarrow M_{m+1}$ and let $\phi_{n,\infty} : M_n \rightarrow M$ be the natural morphism. An easy induction shows that $\prod_{j=n}^m \delta_j \cdot \text{Coker}(\phi_{n,m}) = 0$ for all $m > n \in \mathbb{N}$. Since $\text{Coker}(\phi_{n,\infty}) = \text{colim}_{m \in \mathbb{N}} \text{Coker}(\phi_{n,m})$ we obtain $a_n \cdot \text{Coker}(\phi_{n,\infty}) = 0$ for all $n \in \mathbb{N}$. Therefore $\varepsilon_n \cdot a_n \cdot \text{Coker}(\phi_{n,\infty} \circ \psi_n) = 0$ for all $n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} \varepsilon_n \cdot a_n = V$, the claim follows. \square

Lemma 2.3.26. *Let $\{M_n ; \phi_n : M_n \rightarrow M_{n+1} \mid n \in \mathbb{N}\}$ be a direct system of A -modules and suppose there exist sequences $\{\varepsilon_n \mid n \in \mathbb{N}\}$ and $\{\delta_n \mid n \in \mathbb{N}\}$ of ideals of V such that*

- i) $\lim_{n \rightarrow \infty} \varepsilon_n = V$ and $\prod_{j=0}^{\infty} \delta_j$ is a Cauchy product;*
 - ii) $\varepsilon_n \cdot \text{alExt}_A^i(M_n, N) = \delta_n \cdot \text{alExt}_A^i(\text{Coker}(\phi_n), N) = 0$ for all A -modules N , all $i > 0$ and all $n \in \mathbb{N}$;*
 - iii) $\delta_n \cdot \text{Ker}(\phi_n) = 0$ for all $n \in \mathbb{N}$.*
- Then $\text{colim}_{n \in \mathbb{N}} M_n$ is an almost projective A -module.*

Proof. Let $M = \text{colim}_{n \in \mathbb{N}} M_n$. By the above remark 2.3.11(i) it suffices to show that $\text{alExt}_A^i(M, N)$ vanishes for all $i > 0$ and all A -modules N . The maps ϕ_n define a map $\phi : \oplus_n M_n \rightarrow \oplus_n M_n$ such that we have a short exact sequence $0 \rightarrow \oplus_n M_n \xrightarrow{1-\phi} \oplus_n M_n \rightarrow M \rightarrow 0$. Applying the long exact alExt sequence one obtains a short exact sequence (cp. [22] (3.5.10))

$$0 \rightarrow \lim_{n \in \mathbb{N}} {}^1\text{alExt}_A^{i-1}(M_n, N) \rightarrow \text{alExt}_A^i(M, N) \rightarrow \lim_{n \in \mathbb{N}} \text{alExt}_A^i(M_n, N) \rightarrow 0.$$

Then lemma 2.3.2(ii) implies that $\text{alExt}_A^i(M, N) \simeq 0$ for all $i > 1$ and moreover $\text{alExt}_A^1(M, N)$ is isomorphic to $\lim_{n \in \mathbb{N}} {}^1\text{alHom}_A(M_n, N)$. Let

$$\phi_n^* : \text{alHom}_A(M_{n+1}, N) \rightarrow \text{alHom}_A(M_n, N) \quad f \mapsto f \circ \phi_n$$

be the transpose of ϕ_n and write ϕ_n as a composition $M_n \xrightarrow{p_n} \text{Im}(\phi_n) \xrightarrow{q_n} M_{n+1}$, so that $\phi_n^* = q_n^* \circ p_n^*$, the composition of the respective transposed morphisms. We have monomorphisms

$$\begin{aligned} \text{Coker}(p_n^*) &\hookrightarrow \text{alHom}_A(\text{Ker}(\phi_n), N) \\ \text{Coker}(q_n^*) &\hookrightarrow \text{alExt}_A^1(\text{Coker}(\phi_n), N) \end{aligned}$$

for all $n \in \mathbb{N}$. Hence $\delta_n^2 \cdot \text{Coker}(\phi_n^*) = 0$ for all $n \in \mathbb{N}$. Since $\prod_{n=0}^{\infty} \delta_n^2$ is a Cauchy product, lemma 2.3.2(iii) shows that $\lim_{n \in \mathbb{N}} {}^1\text{alHom}_A(M_n, N) \simeq 0$ and the assertion follows. \square

Proposition 2.3.27. *Suppose that \tilde{m} is a flat V -module. Then for any V -algebra R the functor $M \mapsto M_{\dagger}$ commutes with tensor products and takes flat R^a -modules to flat R -modules.*

Proof. Let M be a flat R^a -module and $N \hookrightarrow N'$ an injective map of R -modules. Denote by K the kernel of the induced map $M_{\dagger} \otimes_R N \rightarrow M_{\dagger} \otimes_R N'$; we have $K^a \simeq 0$. We obtain an exact sequence $0 \rightarrow \tilde{m} \otimes_V K \rightarrow \tilde{m} \otimes_V M_{\dagger} \otimes_R N \rightarrow \tilde{m} \otimes_V M_{\dagger} \otimes_R N'$. But one sees easily that $\tilde{m} \otimes_V K = 0$ and $\tilde{m} \otimes_V M_{\dagger} \simeq M_{\dagger}$, which shows that M_{\dagger} is a flat R -module. Similarly, let M, N be two R^a -modules. Then the natural map $M_* \otimes_R N_* \rightarrow (M \otimes_{R^a} N)_*$ is an almost isomorphism and the assertion follows from remark 2.1.3(i). \square

2.4. Almost homotopical algebra. The formalism of abelian tensor categories provides a minimal framework wherein the rudiments of deformation theory can be developed.

Let $(\mathcal{C}, \otimes, U)$ be an abelian tensor category; we assume henceforth that \otimes is a right exact functor. Let A be a given \mathcal{C} -monoid. A *two-sided ideal* of A is an A -sub-bimodule $I \rightarrow A$. The quotient A/I in the underlying abelian category \mathcal{C} has a unique \mathcal{C} -monoid structure such that $A \rightarrow A/I$ is a morphism of monoids. A/I is unitary if A is. For I, J subobjects of A one denotes $IJ = \text{Im}(I \otimes J \rightarrow A \otimes A \xrightarrow{\mu_A} A)$. If I is a two-sided ideal of A such that $I^2 = 0$, then, using the right exactness of \otimes one checks that I has a natural structure of an A/I -bimodule, unitary when A is.

Definition 2.4.1. A \mathcal{C} -extension of a \mathcal{C} -monoid B by a B -bimodule I is a short exact sequence of objects of \mathcal{C}

$$(2.4.2) \quad X : \quad 0 \longrightarrow I \longrightarrow C \xrightarrow{p} B \longrightarrow 0$$

such that C is a \mathcal{C} -monoid, p is a morphism of \mathcal{C} -monoids, I is a square zero two-sided ideal in C and the E/I -bimodule structure on I coincides with the given bimodule structure on I . The \mathcal{C} -extensions form a category $\mathbf{Exmon}_{\mathcal{C}}$. The morphisms are commutative diagrams with exact rows

$$\begin{array}{ccccccc} X : & 0 & \longrightarrow & I & \longrightarrow & E & \xrightarrow{p} B \longrightarrow 0 \\ & \downarrow & & \downarrow f & & \downarrow g & \downarrow h \\ X' : & 0 & \longrightarrow & I' & \longrightarrow & E' & \xrightarrow{p'} B' \longrightarrow 0 \end{array}$$

such that g and h are morphisms of \mathcal{C} -monoids. We let $\mathbf{Exmon}_{\mathcal{C}}(B, I)$ be the subcategory of $\mathbf{Exmon}_{\mathcal{C}}$ consisting of all \mathcal{C} -extensions of B by I , where the morphisms are all short exact sequences as above such that $f = \mathbf{1}_I$ and $h = \mathbf{1}_B$.

We have also the variant in which all the \mathcal{C} -monoids in (2.4.2) are required to be unitary (resp. to be algebras) and I is a unitary B -bimodule (resp. whose left and right B -module actions coincide, *i.e.* are switched by composition with the “commutativity constraints” $\eta_{B|I}$ and $\eta_{I|B}$, see 2.2); we will call $\mathbf{Exun}_{\mathcal{C}}$ (resp. $\mathbf{Exal}_{\mathcal{C}}$) the corresponding category. For a morphism $\phi : C \rightarrow B$ of \mathcal{C} -monoids, and a \mathcal{C} -extension X in $\mathbf{Exmon}_{\mathcal{C}}(B, I)$, we can pullback X via ϕ to obtain an exact sequence $X * \phi$ with a morphism $\phi^* : X * \phi \rightarrow X$; one checks easily that there exists a unique structure of \mathcal{C} -extension on $X * \phi$ such that ϕ^* is a morphism of \mathcal{C} -extension; then $X * \phi$ is an object in $\mathbf{Exmon}_{\mathcal{C}}(C, I)$. Similarly, given a B -linear morphism $\psi : I \rightarrow J$, we can push out X and obtain a well defined object $\psi * X$ in $\mathbf{Exmon}_{\mathcal{C}}(B, J)$ with a morphism $X \rightarrow \psi * X$ of $\mathbf{Exmon}_{\mathcal{C}}$. In particular, if I_1 and I_2 are two B -bimodules, the functors p_i^* ($i = 1, 2$) associated to the natural projections $p_i : I_1 \oplus I_2 \rightarrow I_i$ establish an equivalence of categories

$$(2.4.3) \quad \mathbf{Exmon}_{\mathcal{C}}(B, I_1 \oplus I_2) \xrightarrow{\sim} \mathbf{Exmon}_{\mathcal{C}}(B, I_1) \times \mathbf{Exmon}_{\mathcal{C}}(B, I_2)$$

whose essential inverse is given by $(E_1, E_2) \mapsto (E_1 \oplus E_2) * \delta$, where $\delta : B \rightarrow B \oplus B$ is the diagonal morphism. A similar statement holds for \mathbf{Exal} and \mathbf{Exun} . These operations can be used to induce an abelian group structure on the set $\mathbf{Exmon}_{\mathcal{C}}(B, I)$ of isomorphism classes of objects of $\mathbf{Exmon}_{\mathcal{C}}(B, I)$ as follows. For any two objects X, Y of $\mathbf{Exmon}_{\mathcal{C}}(B, I)$ we can form $X \oplus Y$ which is an object of $\mathbf{Exmon}_{\mathcal{C}}(B \oplus B, I \oplus I)$. Let $\alpha : I \oplus I \rightarrow I$ be the addition morphism of I . Then we set $X + Y = \alpha * (X \oplus Y) * \delta$. One can check that $X + Y \simeq Y + X$ for any X, Y and that the trivial split \mathcal{C} -extension $B \oplus I$ is a neutral element for $+$. Moreover every isomorphism class has an inverse $-X$. The functors $X \mapsto X * \phi$ and $X \mapsto \psi * X$ commute with the operation thus defined, and induce group homomorphisms

$$\begin{aligned} * \phi &: \mathbf{Exmon}_{\mathcal{C}}(B, I) \rightarrow \mathbf{Exmon}_{\mathcal{C}}(C, I) \\ \psi * &: \mathbf{Exmon}_{\mathcal{C}}(B, I) \rightarrow \mathbf{Exmon}_{\mathcal{C}}(B, J). \end{aligned}$$

We will need the variant $\mathbf{Exal}_{\mathcal{C}}(B, I)$ defined in the same way, starting from $\mathbf{Exal}_{\mathcal{C}}(B, I)$. For instance, if A is an almost algebra (resp. a commutative ring), we can consider the abelian tensor category $\mathcal{C} = A\text{-Mod}$. In this case the \mathcal{C} -extensions will be called simply A -extensions, and we will write \mathbf{Exal}_A rather than $\mathbf{Exal}_{\mathcal{C}}$. In fact the commutative unitary case will soon become prominent in our work, and the more general setup is only required for technical reasons, in the proof of proposition 2.4.6 below, which is the abstract version of a well-known result on the lifting of idempotents over nilpotent ring extensions.

Let A be a \mathcal{C} -monoid. We form the biproduct $A^\dagger = U \oplus A$ in \mathcal{C} . We denote by p_1, p_2 the associated projections from A^\dagger onto U and respectively A . Also, let i_1, i_2 be the natural monomorphisms from U , resp. A to A^\dagger . A^\dagger is equipped with a unitary monoid structure

$$\mu^\dagger = i_2 \circ \mu \circ (p_2 \otimes p_2) + i_2 \circ \ell_A^{-1} \circ (p_1 \otimes p_2) + i_2 \circ r_A^{-1} \circ (p_2 \otimes p_1) + i_1 \circ u^{-1} \circ (p_1 \otimes p_1)$$

where ℓ_A, r_A are the natural isomorphisms provided by [4] (Prop. 1.3) and $u : U \rightarrow U \otimes U$ is as in *loc. cit.* §1. In terms of the ring $A_*^\dagger \simeq U_* \oplus A_*$ this is the multiplication $(u_1, b_1) \cdot (u_2, b_2) = (u_1 \cdot u_2, b_1 \cdot b_2 + b_1 \cdot u_2 + u_1 \cdot b_2)$. Then i_2 is a morphism of monoids and one verifies that the “restriction of scalars” functor i_2^* defines an equivalence from the category $A^\dagger\text{-Uni.Mod}$ of unitary A^\dagger -modules to the category $A\text{-Mod}$ of all A -modules; let j denote the inverse functor. A similar discussion applies to bimodules. Similarly, we derive equivalences of categories

$$\mathbf{Exun}_{\mathcal{C}}(A^\dagger, j(M)) \xrightleftharpoons[\begin{smallmatrix} (-)^\dagger \end{smallmatrix}]{*i_2} \mathbf{Exmon}_{\mathcal{C}}(A, M)$$

for all A -bimodules M .

Next we specialise to $A = U$: for a given U -module M let $e_M = \sigma_M \circ \ell_M : M \rightarrow M$; working out the definitions one finds that the condition that this is a module structure is equivalent to $e_M^2 = e_M$. Let $U \times U$ be the product of U by itself in the category of \mathcal{C} -monoids. There is an isomorphism of unitary \mathcal{C} -monoids $\zeta : U^\dagger \rightarrow U \times U$ given by $\zeta = i_1 \circ p_1 + i_2 \circ p_1 + i_2 \circ p_2$. Another isomorphism is $\tau \circ \zeta$, where τ is the flip $i_1 \circ p_2 + i_2 \circ p_1$. Hence we get equivalences of categories

$$U\text{-Mod} \xrightleftharpoons[\begin{smallmatrix} i_2^* \end{smallmatrix}]{j} U^\dagger\text{-Uni.Mod} \xrightleftharpoons[\begin{smallmatrix} (\tau \circ \zeta)^* \end{smallmatrix}]{(\zeta^{-1})^*} (U \times U)\text{-Uni.Mod}$$

The composition $i_2^* \circ (\zeta^{-1} \circ \tau \circ \zeta)^* \circ j$ defines a self-equivalence of $U\text{-Mod}$ which associates to a given U -module M the new U -module M^{flip} whose underlying object in \mathcal{C} is M and such that $e_{M^{\text{flip}}} = \mathbf{1}_M - e_M$. The same construction applies to U -bimodules and finally we get equivalences

$$(2.4.4) \quad \mathbf{Exmon}_{\mathcal{C}}(U, M) \xrightarrow{\sim} \mathbf{Exmon}_{\mathcal{C}}(U, M^{\text{flip}}) \quad X \mapsto X^{\text{flip}}$$

for all U -bimodules M . If $X = (0 \rightarrow M \rightarrow E \xrightarrow{\pi} U \rightarrow 0)$ is an extension and $X^{\text{flip}} = (0 \rightarrow M^{\text{flip}} \rightarrow E^{\text{flip}} \rightarrow U \rightarrow 0)$, then one verifies that there is a natural isomorphism $X^{\text{flip}} \rightarrow X$ of complexes in \mathcal{C} inducing $-\mathbf{1}_M$ on M , the identity on U and carrying the multiplication morphism on E^{flip} to

$$-\mu_E + \ell_E^{-1} \circ (\pi \otimes \mathbf{1}_E) + r_E^{-1} \circ (\mathbf{1}_E \otimes \pi) : E \otimes E \rightarrow E.$$

In terms of the associated rings, this corresponds to replacing the given multiplication $(x, y) \mapsto x \cdot y$ of E_* by the new operation $(x, y) \mapsto \pi_*(x) \cdot y + \pi_*(y) \cdot x - x \cdot y$.

Lemma 2.4.5. *If M is a U -bimodule whose left and right actions coincide, then every extension of U by M splits uniquely.*

Proof. Using the idempotent e_M we get a U -linear decomposition $M \simeq M_1 \oplus M_2$ where the bimodule structure on M_1 is given by the zero morphisms and the bimodule structure on M_2 is given by ℓ_M^{-1} and r_M^{-1} . We have to prove that $\mathbf{Exmon}_{\mathcal{C}}(U, M)$ is equivalent to a one-point category. By (2.4.3) we can assume that $M = M_1$ or $M = M_2$. By (2.4.4) we have $\mathbf{Exmon}_{\mathcal{C}}(U, M_2) \simeq \mathbf{Exmon}_{\mathcal{C}}(U, M_2^{\text{flip}})$ and on M_2^{flip} the bimodule actions are the zero morphisms. So it is enough to consider $M = M_1$. In this case, if $X = (0 \rightarrow M \rightarrow E \rightarrow U \rightarrow 0)$ is any extension, $\mu_E : E \otimes E \rightarrow E$ factors through a morphism $U \otimes U \rightarrow E$ and composing with $u : U \rightarrow U \otimes U$ we get a right inverse of $E \rightarrow U$, which shows that X is the split extension. Then it is easy to see that X does not have any non-trivial automorphisms, which proves the assertion. \square

Proposition 2.4.6. *i) Let $X = (0 \rightarrow I \rightarrow A \xrightarrow{p} A' \rightarrow 0)$ be a \mathcal{C} -extension and suppose that $e' \in A'_*$ is an idempotent element whose left action on the A' -bimodule I coincides with its right action. Then there exists a unique idempotent $e \in A_*$ such that $p_*(e) = e'$.*

ii) Especially, if A' is unitary and I is a unitary A' -bimodule, then every extension of A' by I is unitary.

Proof. (i) : the hypothesis $e'^2 = e'$ implies that $e' : U \rightarrow A'$ is a morphism of (non-unitary) \mathcal{C} -monoids. We can then replace X by $X * e'$ and thereby assume that $A' = U$, $p : A \rightarrow U$ and I is a (non-unitary) U -bimodule and the right and left actions on I coincide. The assertion to prove is that $\underline{1}_U$ lifts to a unique idempotent $e \in A_*$. However, this follows easily from lemma 2.4.5. To show (ii), we observe that, by (i), the unit $\underline{1}_{A'}$ of A'_* lifts uniquely to an idempotent $e \in A_*$. We have to show that e is a unit for A_* . Let us show the left unit property. Via $e : U \rightarrow A$ we can view the extension X as an exact sequence of left U -modules. We can then split X as the direct sum $X_1 \oplus X_2$ where X_1 is a sequence of unitary U -modules and X_2 is a sequence of U -modules with trivial actions. But by hypothesis, on I and on A the U -module structure is unitary, so $X = X_1$ and this is the left unit property. \square

So much for the general nonsense; we now return to almost algebras. As already announced, from here on, we assume throughout that $\tilde{\mathfrak{m}}$ is a flat V -module. As an immediate consequence of proposition 2.4.6 we get natural equivalences of categories

$$(2.4.7) \quad \mathbf{Exal}_{A_1}(B_1, M_1) \times \mathbf{Exal}_{A_2}(B_2, M_2) \xrightarrow{\sim} \mathbf{Exal}_{A_1 \times A_2}(B_1 \times B_2, M_1 \oplus M_2)$$

whenever A_1, A_2 are V^a -algebras, B_i is a A_i -algebra and M_i is a (unitary) B_i -module, $i = 1, 2$.

Notice that, if $A = R^a$ for some V -algebra R , S (resp. J) is a R -algebra (resp. an S -module) and X is any object of $\mathbf{Exal}_R(S, J)$, then by applying termwise the localisation functor we get an object X^a of $\mathbf{Exal}_A(S^a, J^a)$. With this notation we have the following lemma.

Lemma 2.4.8. i) Let B be any A -algebra and I a B -module. The natural functor

$$(2.4.9) \quad \mathbf{Exal}_{A_{!!}}(B_{!!}, I_*) \rightarrow \mathbf{Exal}_A(B, I) \quad X \mapsto X^a$$

is an equivalence of categories.

ii) The equivalence (2.4.9) induces a group isomorphism $\mathbf{Exal}_{A_{!!}}(B_{!!}, I_*) \xrightarrow{\sim} \mathbf{Exal}_A(B, I)$ functorial in all arguments.

Proof. Of course (ii) is an immediate consequence of (i). To show (i), let $X = (0 \rightarrow I \rightarrow E \rightarrow B \rightarrow 0)$ be any object of $\mathbf{Exal}_A(B, I)$. Using corollary 2.2.10 one sees easily that the sequence $X_! = (0 \rightarrow I_! \rightarrow E_{!!} \rightarrow B_{!!} \rightarrow 0)$ is right exact; $X_!$ won't be exact in general, unless B (and therefore E) is an exact algebra. In any case, the kernel of $I_! \rightarrow E_{!!}$ is almost zero, so we get an extension of $B_{!!}$ by a quotient of $I_!$ which maps to I_* . In particular we get by pushout an extension $X_{!*}$ by I_* , i.e. an object of $\mathbf{Exal}_{A_{!!}}(B_{!!}, I_*)$ and in fact the assignment $X \mapsto X_{!*}$ is an essential inverse for the functor (2.4.9). \square

Remark 2.4.10. By inspecting the proof, we see that one can replace I_* by $I_{!*} = \text{Im}(I_! \rightarrow I_*)$ in (i) and (ii) above. When B is exact, also $I_!$ will do.

In [14] (II.1.2) it is shown how to associate to any ring homomorphism $R \rightarrow S$ a natural simplicial complex of S -modules denoted $\mathbb{L}_{S/R}$ and called the cotangent complex of S over R .

Definition 2.4.11. Let $A \rightarrow B$ be a morphism of almost V -algebras. The *almost cotangent complex* of B over A is the simplicial $B_{!!}$ -module

$$\mathbb{L}_{B/A} = B_{!!} \otimes_{(V^a \times B)_{!!}} \mathbb{L}_{(V^a \times B)_{!!}/(V^a \times A)_{!!}}.$$

Usually we will want to view $\mathbb{L}_{B/A}$ as an object of the derived category $\mathbf{D}_\bullet(s.B_{!!})$ of simplicial $B_{!!}$ -modules. Indeed, the hyperext functors computed in this category relate the cotangent complex to a number of important invariants. Recall that, for any simplicial ring R and any two R -modules E, F the hyperext of E and F is the abelian group defined as

$$\mathbb{E}xt_R^p(E, F) = \text{colim}_{n \geq -p} \text{Hom}_{\mathbf{D}_\bullet(R)}(\sigma^n E, \sigma^{n+p} F)$$

(where σ is the suspension functor of [14] (I.3.2.1.4)).

Let us fix an almost algebra A . First we want to establish the relationship with differentials.

Definition 2.4.12. Let B be any A -algebra, M any B -module.

i) An A -derivation of B with values in M is an A -linear morphism $\partial : B \rightarrow M$ such that $\partial(b_1 \cdot b_2) = b_1 \cdot \partial(b_2) + b_2 \cdot \partial(b_1)$ for $b_1, b_2 \in B_*$. The set of all M -valued A -derivations of B forms a V -module $\text{Der}_A(B, M)$ and the almost V -module $\text{Der}_A(B, M)^a$ has a natural structure of B -module.

ii) We reserve the notation $I_{B/A}$ for the ideal $\text{Ker}(\mu_{B/A} : B \otimes_A B \rightarrow B)$. The *module of relative differentials* of ϕ is defined as the (left) B -module $\Omega_{B/A} = I_{B/A}/I_{B/A}^2$. It is endowed with a natural A -derivation $\delta : B \rightarrow \Omega_{B/A}$ defined by $b \mapsto \underline{1} \otimes b - b \otimes \underline{1}$ for all $b \in B_*$. The assignment $(A \rightarrow B) \mapsto \Omega_{B/A}$ defines a functor

$$\Omega : V^a\text{-Alg.Morph} \rightarrow V^a\text{-Alg.Mod}$$

from the category of morphisms $A \rightarrow B$ of almost V -algebras to the category $V^a\text{-Alg.Mod}$ consisting of all pairs (B, M) where B is an almost V -algebra and M is a B -module. The morphisms in $V^a\text{-Alg.Morph}$ are the commutative squares; the morphisms $(B, M) \rightarrow (B', M')$ in $V^a\text{-Alg.Mod}$ are all pairs (ϕ, f) where $\phi : B \rightarrow B'$ is a morphism of almost V -algebras and $f : B' \otimes_B M \rightarrow M'$ is a morphism of B' -modules.

The module of relative differentials enjoys the familiar universal properties that one expects. In particular $\Omega_{B/A}$ represents the functor $\text{Der}_A(B, -)$, i.e. for any (left) B -module M the morphism

$$(2.4.13) \quad \text{Hom}_B(\Omega_{B/A}, M) \rightarrow \text{Der}_A(B, M) \quad f \mapsto f \circ \delta$$

is an isomorphism. As an exercise, the reader can supply the proof for this claim and for the following standard proposition.

Proposition 2.4.14. i) Let B and C be two A -algebras. Then there is a natural isomorphism:

$$\Omega_{C \otimes_A B/C} \simeq C \otimes_A \Omega_{B/A}.$$

ii) Let B be an A -algebra, C a B -algebra. There is a natural exact sequence of C -modules:

$$C \otimes_B \Omega_{B/A} \rightarrow \Omega_{C/A} \rightarrow \Omega_{C/B} \rightarrow 0.$$

iii) Let I be an ideal of the A -algebra B and let $C = B/I$ be the quotient A -algebra. Then there is a natural exact sequence: $I/I^2 \rightarrow C \otimes_B \Omega_{B/A} \rightarrow \Omega_{C/A} \rightarrow 0$.

iv) The functor $\Omega : V^a\text{-Alg.Morph} \rightarrow V^a\text{-Alg.Mod}$ commutes with all colimits. \square

Lemma 2.4.15. For any A -algebra B there is a natural isomorphism of $B_{!!}$ -modules

$$(\Omega_{B/A})_! \simeq \Omega_{B_{!!}/A_{!!}}.$$

Proof. Using the adjunction (2.4.13) we are reduced to showing that the natural map

$$\phi_M : \text{Der}_{A_{!!}}(B_{!!}, M) \rightarrow \text{Der}_A(B, M^a)$$

is a bijection for all $B_{!!}$ -modules M . Given $\partial : B \rightarrow M^a$ we construct $\partial_! : B_! \rightarrow M_!^a \rightarrow M$. We extend $\partial_!$ to $V \oplus B_!$ by setting it equal to zero on V . Then it is easy to check that the resulting map descends to $B_{!!}$, hence giving an A -derivation $B_{!!} \rightarrow M$. This procedure yields a right inverse ψ_M to ϕ_M . To show that ϕ_M is injective, suppose that $\partial : B_{!!} \rightarrow M$ is an almost zero A -derivation. Composing with the natural A -linear map $B_! \rightarrow B_{!!}$ we obtain an almost zero map $\partial' : B_! \rightarrow M$. But $\mathfrak{m} \cdot B_! = B_!$, hence $\partial' = 0$. This implies that in fact $\partial = 0$, and the assertion follows. \square

Proposition 2.4.16. Let M be a B -module. There exists a natural isomorphism of $B_{!!}$ -modules

$$\text{Ext}_{B_{!!}}^0(\mathbb{L}_{B/A}, M_!) \simeq \text{Der}_A(B, M).$$

Proof. To ease notation, set $\tilde{A} = V^a \times A$ and $\tilde{B} = V^a \times B$. We have natural isomorphisms :

$$\begin{aligned} \mathbb{E}xt_{B_{!!}}^0(\mathbb{L}_{B/A}, M_!) &\simeq \mathbb{E}xt_{\tilde{B}_{!!}}^0(\mathbb{L}_{\tilde{B}_{!!}/\tilde{A}_{!!}}, M_!) && \text{by [14] (I.3.3.4.4)} \\ &\simeq \text{Der}_{\tilde{A}_{!!}}(\tilde{B}_{!!}, M_!) && \text{by [14] (II.1.2.4.2)} \\ &\simeq \text{Der}_{\tilde{A}}(\tilde{B}, M) && \text{by lemma 2.4.15.} \end{aligned}$$

But it is easy to see that the natural map $\text{Der}_A(B, M) \rightarrow \text{Der}_{\tilde{A}}(\tilde{B}, M)$ is an isomorphism. \square

Theorem 2.4.17. *There is a natural isomorphism*

$$\text{Exal}_A(B, M) \rightarrow \mathbb{E}xt_{B_{!!}}^1(\mathbb{L}_{B/A}, M_!).$$

Proof. With the notation of the proof of proposition 2.4.16 we have natural isomorphisms

$$\begin{aligned} \mathbb{E}xt_{B_{!!}}^1(\mathbb{L}_{B/A}, M_!) &\simeq \mathbb{E}xt_{\tilde{B}_{!!}}^1(\mathbb{L}_{\tilde{B}_{!!}/\tilde{A}_{!!}}, M_!) && \text{by [14] (I.3.3.4.4)} \\ &\simeq \text{Exal}_{\tilde{A}_{!!}}(\tilde{B}_{!!}, M_!) && \text{by [14] (III.1.2.3)} \\ &\simeq \text{Exal}_{\tilde{A}}(\tilde{B}, M) \end{aligned}$$

where the last isomorphism follows directly from lemma 2.4.8(ii) and the subsequent remark 2.4.10. Finally, (2.4.7) shows that $\text{Exal}_{\tilde{A}}(\tilde{B}, M) \simeq \text{Exal}_A(B, M)$, as required. \square

Moreover we have the following transitivity theorem as in [14] (II.2.1.2).

Theorem 2.4.18. *Let $A \rightarrow B \rightarrow C$ be a sequence of morphisms of almost V -algebras. There exists a natural distinguished triangle of $\mathbf{D}_\bullet(s.C_{!!})$*

$$C_{!!} \otimes_{B_{!!}} \mathbb{L}_{B/A} \xrightarrow{u} \mathbb{L}_{C/A} \xrightarrow{v} \mathbb{L}_{C/B} \longrightarrow C_{!!} \otimes_{B_{!!}} \sigma \mathbb{L}_{B/A}$$

where the morphisms u and v are obtained by functoriality of \mathbb{L} .

Proof. It follows directly from *loc. cit.* \square

Proposition 2.4.19. *Let $(A_\lambda \rightarrow B_\lambda)_{\lambda \in I}$ be a system of almost V -algebra morphisms indexed by a small filtered category I . Then there is a natural isomorphism in $\mathbf{D}_\bullet(s.\text{colim}_{\lambda \in I} B_{\lambda!!})$*

$$\text{colim}_{\lambda \in I} \mathbb{L}_{B_\lambda/A_\lambda} \simeq \mathbb{L}_{\text{colim}_{\lambda \in I} B_\lambda / \text{colim}_{\lambda \in I} A_\lambda}.$$

Proof. Remark 2.2.15 gives an isomorphism : $\text{colim}_{\lambda \in I} A_{\lambda!!} \xrightarrow{\sim} (\text{colim}_{\lambda \in I} A_\lambda)_{!!}$ (and likewise for $\text{colim}_{\lambda \in I} B_\lambda$). Then the claim follows from [14] (II.1.2.3.4). \square

Next we want to prove the almost version of the flat base change theorem [14] (II.2.2.1). To this purpose we need some preparation.

Proposition 2.4.20. *Let B and C be two A -algebras and set $T_i = \text{Tor}_i^{A_{!!}}(B_{!!}, C_{!!})$. If A , B , C and $B \otimes_A C$ are all exact, then for every $i > 0$ the natural morphism $\tilde{\mathfrak{m}} \otimes_V T_i \rightarrow T_i$ is an isomorphism.*

Proof. For any almost V -algebra D we let k_D denote the complex of $D_{!!}$ -modules $[\tilde{\mathfrak{m}} \otimes_V D_{!!} \rightarrow D_{!!}]$ placed in degrees $-1, 0$; we have a distinguished triangle

$$\mathcal{A}(D) : \tilde{\mathfrak{m}} \otimes_V D_{!!} \longrightarrow D_{!!} \longrightarrow k_D \longrightarrow \tilde{\mathfrak{m}} \otimes_V D_{!!}[1].$$

By the assumption, the natural map $k_A \rightarrow k_B$ is a quasi-isomorphism and $\tilde{\mathfrak{m}} \otimes_V B_{!!} \simeq B_!$. On the other hand, for all $i \in \mathbb{N}$ we have

$$\text{Tor}_i^{A_{!!}}(k_B, C_{!!}) \simeq \text{Tor}_i^{A_{!!}}(k_A, C_{!!}) \simeq H^{-i}(k_A \otimes_{A_{!!}} C_{!!}) = H^{-i}(k_C).$$

In particular $\text{Tor}_i^{A_{!!}}(k_B, C_{!!}) = 0$ for all $i > 1$. As $\tilde{\mathfrak{m}}$ is flat over V , we have $\tilde{\mathfrak{m}} \otimes_V T_i \simeq \text{Tor}_i^{A_{!!}}(\tilde{\mathfrak{m}} \otimes_V B_{!!}, C_{!!})$. Then by the long exact Tor sequence associated to $\mathcal{A}(B) \overset{\mathbf{L}}{\otimes}_{A_{!!}} C_{!!}$ we

get the assertion for all $i > 1$. Next we consider the natural map of distinguished triangles $\mathcal{A}(A) \otimes_{A!!}^{\mathbf{L}} A!! \rightarrow \mathcal{A}(B) \otimes_{A!!}^{\mathbf{L}} C!!$; writing down the associated morphism of long exact Tor sequences, we obtain a diagram with exact rows :

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Tor}_1^{A!!}(k_A, A!!) & \xrightarrow{\partial} & (\tilde{\mathfrak{m}} \otimes_V A!!) \otimes_{A!!} A!! & \xrightarrow{i} & A!! \otimes_{A!!} A!! \\ & \downarrow & & \downarrow & & \downarrow \\ & \mathrm{Tor}_1^{A!!}(k_B, C!!) & \xrightarrow{\partial'} & (\tilde{\mathfrak{m}} \otimes_V B!!) \otimes_{A!!} C!! & \xrightarrow{i'} & B!! \otimes_{A!!} C!! \end{array}$$

By the above, the leftmost vertical map is an isomorphism; moreover, the assumption gives $\mathrm{Ker}(i) \simeq \mathrm{Ker}(\tilde{\mathfrak{m}} \rightarrow V) \simeq \mathrm{Ker}(i')$. Then, since ∂ is injective, also ∂' must be injective, which implies our assertion for the remaining case $i = 1$. \square

Corollary 2.4.21. *Keep the notation of proposition 2.4.20 and suppose that $\mathrm{Tor}_i^A(B, C) \simeq 0$ for some $i > 0$. Then the corresponding T_i vanishes.* \square

Theorem 2.4.22. *Let B, A' be two A -algebras. Suppose that the natural morphism $B \otimes_A^{\mathbf{L}} A' \rightarrow B' = B \otimes_A A'$ is an isomorphism in $\mathbf{D}_\bullet(s.A)$. Then the natural morphisms*

$$\begin{aligned} B'_{!!} \otimes_{B!!} \mathbb{L}_{B/A} &\longrightarrow \mathbb{L}_{B'/A'} \\ (B'_{!!} \otimes_{B!!} \mathbb{L}_{B/A}) \oplus (B'_{!!} \otimes_{A'!!} \mathbb{L}_{A'/A}) &\longrightarrow \mathbb{L}_{B'/A} \end{aligned}$$

are quasi-isomorphisms.

Proof. Let us remark that the functor $D \mapsto V^a \times D : A\text{-}\mathbf{Alg} \rightarrow (V^a \times A)\text{-}\mathbf{Alg}$ commutes with tensor products; hence the same holds for the functor $D \mapsto (V^a \times D)_{!!}$ (see remark 2.2.15). Then, in view of corollary 2.4.21, the theorem is reduced immediately to [14] (II.2.2.1). \square

As an application we obtain the vanishing of the almost cotangent complex for a certain class of morphisms.

Theorem 2.4.23. *Let $R \rightarrow S$ be a morphism of almost algebras such that*

$$\mathrm{Tor}_i^R(S, S) \simeq 0 \simeq \mathrm{Tor}_i^{S \otimes_R S}(S, S) \quad \text{for all } i > 0$$

(for the natural $S \otimes_R S$ -module structure induced by $\mu_{S/R}$). Then $\mathbb{L}_{S/R} \simeq 0$ in $\mathbf{D}_\bullet(S_{!!})$.

Proof. Since $\mathrm{Tor}_i^R(S, S) = 0$ for all $i > 0$, theorem 2.4.22 applies (with $A = R$ and $B = A' = S$), giving the natural isomorphisms

$$(2.4.24) \quad \begin{aligned} (S \otimes_R S)_{!!} \otimes_{S_{!!}} \mathbb{L}_{S/R} &\simeq \mathbb{L}_{S \otimes_R S/S} \\ ((S \otimes_R S)_{!!} \otimes_{S_{!!}} \mathbb{L}_{S/R}) \oplus ((S \otimes_R S)_{!!} \otimes_{S_{!!}} \mathbb{L}_{S/R}) &\simeq \mathbb{L}_{S \otimes_R S/R} \end{aligned}$$

Since $\mathrm{Tor}_i^{S \otimes_R S}(S, S) = 0$, the same theorem also applies with $A = S \otimes_R S$, $B = S$, $A' = S$, and we notice that in this case $B' \simeq S$; hence we have

$$(2.4.25) \quad \mathbb{L}_{S/S \otimes_R S} \simeq S_{!!} \otimes_{S_{!!}} \mathbb{L}_{S/S \otimes_R S} \simeq \mathbb{L}_{S/S} \simeq 0.$$

Next we apply transitivity to the sequence $R \rightarrow S \otimes_R S \rightarrow S$, to obtain (thanks to (2.4.25))

$$(2.4.26) \quad S_{!!} \otimes_{S \otimes_R S, !!} \mathbb{L}_{S \otimes_R S/R} \simeq \mathbb{L}_{S/R}.$$

Applying $S_{!!} \otimes_{S \otimes_R S, !!} -$ to the second isomorphism (2.4.24) we obtain

$$(2.4.27) \quad \mathbb{L}_{S/R} \oplus \mathbb{L}_{S/R} \simeq S_{!!} \otimes_{S \otimes_R S, !!} \mathbb{L}_{S \otimes_R S/R}.$$

Finally, composing (2.4.26) and (2.4.27) we derive

$$(2.4.28) \quad \mathbb{L}_{S/R} \oplus \mathbb{L}_{S/R} \xrightarrow{\sim} \mathbb{L}_{S/R}.$$

However, by inspection, the isomorphism (2.4.28) is the sum map. Consequently $\mathbb{L}_{S/R} \simeq 0$, as claimed. \square

Finally we have a fundamental spectral sequence as in [14] (III.3.3.2).

Theorem 2.4.29. *Let $\phi : A \rightarrow B$ be a morphism of almost algebras such that $B \otimes_A B \simeq B$ (e.g. such that B is a quotient of A). Then there is a first quadrant homology spectral sequence of bigraded almost algebras*

$$E_{pq}^2 = H_{p+q}(\mathrm{Sym}_B^q(\mathbb{L}_{B/A}^a)) \Rightarrow \mathrm{Tor}_{p+q}^A(B, B).$$

Proof. We replace ϕ by $\mathbf{1}_{V^a} \times \phi$ and apply the functor $B \mapsto B_{!!}$ (which commutes with tensor products by remark 2.2.15) thereby reducing the assertion to the above mentioned [14] (III.3.3.2). \square

3. ALMOST RING THEORY

3.1. Flat, unramified and étale morphisms. Let $A \rightarrow B$ be a morphism of almost V -algebras. Using the natural “multiplication” morphism of A -algebras $\mu_{B/A} : B \otimes_A B \rightarrow B$ we can view B as a $B \otimes_A B$ -algebra.

Definition 3.1.1. Let $\phi : A \rightarrow B$ be a morphism of almost V -algebras.

- i) We say that ϕ is a *flat* (resp. *faithfully flat*, resp. *almost projective*) *morphism* if B is a flat (resp. faithfully flat, resp. almost projective) A -module.
- ii) We say that ϕ is *almost finite* (resp. *finite*) if B is an almost finitely generated (resp. finitely generated) A -module.
- iii) We say that ϕ is *weakly unramified* (resp. *unramified*) if B is a flat (resp. almost projective) $B \otimes_A B$ -module (via the morphism $\mu_{B/A}$ defined above).
- iv) ϕ is *weakly étale* (resp. *étale*) if it is flat and weakly unramified (resp. unramified).

Lemma 3.1.2. *Let $\phi : A \rightarrow B$ and $\psi : B \rightarrow C$ be morphisms of almost V -algebras.*

- i) *Any base change of a flat (resp. almost projective, resp. faithfully flat, resp. almost finite, resp. weakly unramified, resp. unramified, resp. weakly étale, resp. étale) morphism is flat (resp. almost projective, resp. faithfully flat, resp. almost finite, resp. weakly unramified, resp. unramified, resp. weakly étale, resp. étale);*
- ii) *if both ϕ and ψ are flat (resp. almost projective, resp. faithfully flat, resp. almost finite, resp. weakly unramified, resp. unramified, resp. weakly étale, resp. étale), then so is $\psi \circ \phi$;*
- iii) *if ϕ is flat and $\psi \circ \phi$ is faithfully flat, then ϕ is faithfully flat;*
- iv) *if ϕ is weakly unramified and $\psi \circ \phi$ is flat (resp. weakly étale), then ψ is flat (resp. weakly étale);*
- v) *If ϕ is unramified and $\psi \circ \phi$ is étale, then ψ is étale;*
- vi) *ϕ is faithfully flat if and only if it is a monomorphism and B/A is a flat A -module;*
- vii) *if ϕ is almost finite and weakly unramified, then ϕ is unramified.*

Proof. For (vi) use the Tor sequences. In view of proposition 2.3.15(ii), to show (vii) it suffices to know that B is an almost finitely presented $B \otimes_A B$ -module; but this follows from the existence of an epimorphism of $B \otimes_A B$ -modules $(B \otimes_A B) \otimes_A B \rightarrow \mathrm{Ker}(\mu_{B/A})$ defined by $x \otimes b \mapsto x \cdot (\mathbf{1} \otimes b - b \otimes \mathbf{1})$. Of the remaining assertions, only (iv) and (v) are not obvious, but the proof is just the “almost version” of a well-known argument. Let us show (v); the same argument applies to (iv). We remark that $\mu_{B/A}$ is an étale morphism, since ϕ is unramified. Define $\Gamma_\psi = \mathbf{1}_C \otimes_B \mu_{B/A}$. By (i), Γ_ψ is étale. Define also $p = (\psi \circ \phi) \otimes_A \mathbf{1}_B$. By (i), p is flat (resp. étale). The claim follows by remarking that $\psi = \Gamma_\psi \circ p$ and applying (ii). \square

Remark 3.1.3. i) Suppose we work in the classical limit case, that is, $V = \mathfrak{m}$ (cp. example 2.1.1(ii)). Then we caution the reader that our notion of “étale morphism” is more general than the usual one, as defined in [10]. The relationship between the usual notion and ours is discussed in the digression at the end of section 3.4.

ii) The naive hope that the functor $A \mapsto A_{!!}$ might preserve flatness is crushed by the following counterexample. Let (V, \mathfrak{m}) be as in example 2.1.1(i) and let k be the residue field of V . Consider the flat map $V \times V \rightarrow V$ defined as $(x, y) \mapsto x$. We get a flat morphism $V^a \times V^a \rightarrow V^a$ in $V^a\text{-Alg}$; applying the left adjoint to localisation yields a map $V \times_k V \rightarrow V$ that is not flat. On the other hand, faithful flatness is preserved. Indeed, let $\phi : A \rightarrow B$ be a morphism of almost algebras. Then ϕ is a monomorphism if and only if $\phi_{!!}$ is injective; moreover, $B_{!!}/\text{Im}(A_{!!}) \simeq B_! / A_!$, which is flat over $A_{!!}$ if and only if B/A is flat over A , by proposition 2.3.27.

We will find useful to study certain “almost idempotents”, as in the following proposition.

Proposition 3.1.4. *A morphism $\phi : A \rightarrow B$ is unramified if and only if there exists an almost element $e_{B/A} \in B \otimes_A B_*$ such that*

- i) $e_{B/A}^2 = e_{B/A}$;
- ii) $\mu_{B/A}(e_{B/A}) = \underline{1}$;
- iii) $x \cdot e_{B/A} = 0$ for all $x \in I_{B/A*}$.

Proof. Suppose that ϕ is unramified. We start by showing that for every $\varepsilon \in \mathfrak{m}$ there exist almost elements e_ε of $B \otimes_A B$ such that

$$(3.1.5) \quad e_\varepsilon^2 = \varepsilon \cdot e_\varepsilon \quad \mu_{B/A}(e_\varepsilon) = \varepsilon \cdot \underline{1} \quad I_{B/A*} \cdot e_\varepsilon = 0.$$

Since B is an almost projective $B \otimes_A B$ -module, for every $\varepsilon \in \mathfrak{m}$ there exists an “approximate splitting” for the epimorphism $\mu_{B/A} : B \otimes_A B \rightarrow B$, i.e. a $B \otimes_A B$ -linear morphism $u_\varepsilon : B \rightarrow B \otimes_A B$ such that $\mu_{B/A} \circ u_\varepsilon = \varepsilon \cdot \mathbf{1}_B$. Set $e_\varepsilon = u_\varepsilon \circ \underline{1} : A \rightarrow B \otimes_A B$. We see that $\mu_{B/A}(e_\varepsilon) = \varepsilon \cdot \underline{1}$. To show that $e_\varepsilon^2 = \varepsilon \cdot e_\varepsilon$ we use the $B \otimes_A B$ -linearity of u_ε to compute

$$e_\varepsilon^2 = e_\varepsilon \cdot u_\varepsilon(\underline{1}) = u_\varepsilon(\mu_{B/A}(e_\varepsilon) \cdot \underline{1}) = u_\varepsilon(\mu_{B/A}(e_\varepsilon)) = \varepsilon \cdot e_\varepsilon.$$

Next take any almost element x of $I_{B/A}$ and compute

$$x \cdot e_\varepsilon = x \cdot u_\varepsilon(\underline{1}) = u_\varepsilon(\mu_{B/A}(x) \cdot \underline{1}) = 0.$$

This establishes (3.1.5). Next let us take any other $\delta \in \mathfrak{m}$ and a corresponding almost element e_δ . Both $\varepsilon \cdot \underline{1} - e_\varepsilon$ and $\delta \cdot \underline{1} - e_\delta$ are elements of $I_{B/A*}$, hence we have $(\delta \cdot \underline{1} - e_\delta) \cdot e_\varepsilon = 0 = (\varepsilon \cdot \underline{1} - e_\varepsilon) \cdot e_\delta$ which implies

$$(3.1.6) \quad \delta \cdot e_\varepsilon = \varepsilon \cdot e_\delta \quad \text{for all } \varepsilon, \delta \in \mathfrak{m}.$$

Let us define a map $e_{B/A} : \mathfrak{m} \otimes_V \mathfrak{m} \rightarrow B \otimes_A B_*$ by the rule

$$(3.1.7) \quad \varepsilon \otimes \delta \mapsto \delta \cdot e_\varepsilon \quad \text{for all } \varepsilon, \delta \in \mathfrak{m}.$$

To show that (3.1.7) does indeed determine a well defined morphism, we need to check that $\delta \cdot v \cdot e_\varepsilon = \delta \cdot e_{v \cdot \varepsilon}$ and $\delta \cdot e_{\varepsilon + \varepsilon'} = \delta \cdot (e_\varepsilon + e_{\varepsilon'})$ for all $\varepsilon, \varepsilon', \delta \in \mathfrak{m}$ and all $v \in V$. However, both identities follow easily by a repeated application of (3.1.6). It is easy to see that $e_{B/A}$ defines an almost element with the required properties.

Conversely, suppose an almost element $e_{B/A}$ of $B \otimes_A B$ is given with the stated properties. We define $u : B \rightarrow B \otimes_A B$ by $b \mapsto e_{B/A} \cdot (1 \otimes b)$ ($b \in B_*$) and $v = \mu_{B/A}$. Then (iii) says that u is a $B \otimes_A B$ -linear morphism and (ii) shows that $v \circ u = \mathbf{1}_B$. Hence, by lemma 2.3.12, ϕ is unramified. \square

Corollary 3.1.8. *Under the hypotheses and notation of the proposition, the ideal $I = I_{B/A}$ has a natural structure of A -algebra, with unit morphism given by $\underline{1}_{I/A} = \underline{1}_{B \otimes_A B/A} - e_{B/A}$ and whose multiplication is the restriction of $\mu_{B \otimes_A B/A}$ to I . Moreover the natural morphism*

$$B \otimes_A B \rightarrow I_{B/A} \oplus B \quad x \mapsto (x \cdot \underline{1}_{I/A} \oplus \mu_{B/A}(x))$$

is an isomorphism of A -algebras.

Proof. Left to the reader as an exercise. \square

3.2. Almost traces. Let A be an almost V -algebra. For any integer $n > 0$, the standard direct sum decomposition of A^n determines uniquely A -linear morphisms $A \xrightarrow{e_i^A} A^n \xrightarrow{\pi_j^A} A$ (for $i, j = 1, \dots, n$) such that $\pi_j^A \circ e_i^A = \delta_{ij} \cdot \mathbf{1}_A$ for all i, j and $\sum_{i=1}^n e_i^A \circ \pi_i^A = \mathbf{1}_{A^n}$. We can then define a natural *trace homomorphism*

$$(3.2.1) \quad \text{Tr} : \text{alHom}_A(A^n, A^n) \rightarrow A \quad \phi \mapsto \sum_{i=1}^n \pi_i^A \circ \phi \circ e_i^A$$

which is an A -linear morphism. For any $\phi, \psi \in \text{alHom}_A(A^n, A^n)_*$ we have $\text{Tr}(\phi \circ \psi) = \text{Tr}(\psi \circ \phi)$. It follows easily that Tr is independent of the given direct sum decomposition of A^n . More generally, suppose that M is an almost projective almost finitely generated A -module. Then for any $\varepsilon \in \mathfrak{m}$ we can find $n = n(\varepsilon)$ and morphisms

$$(3.2.2) \quad M \xrightarrow{u_\varepsilon} A^n \xrightarrow{v_\varepsilon} M$$

such that $v_\varepsilon \circ u_\varepsilon = \varepsilon \cdot \mathbf{1}_M$. Let $E(M) = \text{alHom}_A(M, M)$; notice that $E(M)_*$ is naturally isomorphic to $\text{Hom}_A(M, M)$. We consider the A -linear morphism

$$(3.2.3) \quad t_\varepsilon : E(M) \rightarrow A \quad \phi \mapsto \text{Tr}(u_\varepsilon \circ \phi \circ v_\varepsilon) \quad (\phi \in E(M)_*).$$

Now, pick any other $\delta \in \mathfrak{m}$. We compute

$$\begin{aligned} \varepsilon \cdot t_\delta(\phi) &= \varepsilon \cdot \text{Tr}(u_\delta \circ \phi \circ v_\delta) = \text{Tr}(u_\delta \circ v_\varepsilon \circ u_\varepsilon \circ \phi \circ v_\delta) \\ &= \text{Tr}(u_\varepsilon \circ \phi \circ v_\delta \circ u_\delta \circ v_\varepsilon) = \delta \cdot \text{Tr}(u_\varepsilon \circ \phi \circ v_\varepsilon) = \delta \cdot t_\varepsilon(\phi). \end{aligned}$$

This allows us to define a map

$$t_{M/A} : \mathfrak{m} \otimes_V \mathfrak{m} \otimes_V E(M)_* \rightarrow A_*$$

by setting $\varepsilon \otimes \delta \otimes \phi \mapsto \varepsilon \cdot t_\delta(\phi)$. We leave to the reader the verification that $t_{M/A}$ is well defined and does not depend on the choice of t_δ . It induces a morphism $E(M) \rightarrow A$ that we denote again by $t_{M/A}$ and we call the *almost trace morphism* for the almost A -module M .

Let $f \in M_*^*$, $m \in M_*$ and define $\phi_{f,m} \in E(M)_*$ by the formula $\phi_{f,m}(m') = f(m') \cdot m$ for all $m' \in M_*$. We have the following :

Lemma 3.2.4. *With the above notation : $t_{M/A}(\phi_{f,m}) = f(m)$.*

Proof. Let $f : M \rightarrow A$ and $m : A \rightarrow M$ be given. Obviously we have $\phi_{f,m} = m \circ f$ and $f(m) = f \circ m$. Pick morphisms u_ε and v_ε as in (3.2.2). Using the foregoing notation, we can write :

$$\begin{aligned} t_\varepsilon(\phi_{f,m}) &= \sum_{i=1}^n (\pi_i^A \circ u_\varepsilon \circ m) \circ (f \circ v_\varepsilon \circ e_i^A) \\ &= \sum_{i=1}^n (f \circ v_\varepsilon \circ e_i^A) \circ (\pi_i^A \circ u_\varepsilon \circ m) \\ &= f \circ v_\varepsilon \circ u_\varepsilon \circ m = \varepsilon \cdot f \circ m \end{aligned}$$

from which the claim follows directly. \square

Lemma 3.2.5. *Let M and N be almost finitely generated almost projective A -modules, and $\phi : M \rightarrow N$, $\psi : N \rightarrow M$ two A -linear morphisms. Then :*

i) $t_{M/A}(\psi \circ \phi) = t_{N/A}(\phi \circ \psi)$.

ii) *If $\psi \circ \phi = a \cdot \mathbf{1}_M$ and $\phi \circ \psi = a \cdot \mathbf{1}_N$ for some $a \in A_*$, and if, furthermore, there exist $u \in \text{End}_A(M)$, $v \in \text{End}_A(N)$ such that $v \circ \phi = \phi \circ u$, then $a \cdot (t_{M/A}(u) - t_{N/A}(v)) = 0$.*

Proof. (i) is left to the reader as an exercise. For (ii) we compute using (i) : $a \cdot t_{M/A}(u) = t_{M/A}(\psi \circ \phi \circ u) = t_{M/A}(\psi \circ v \circ \phi) = t_{N/A}(v \circ \phi \circ \psi) = a \cdot t_{N/A}(v)$. \square

Proposition 3.2.6. *Let $\underline{M} = (0 \rightarrow M_1 \xrightarrow{i} M_2 \xrightarrow{p} M_3 \rightarrow 0)$ be an exact sequence of almost finitely generated almost projective A -modules, and let $\underline{u} = (u_1, u_2, u_3) : \underline{M} \rightarrow \underline{M}$ be an endomorphism of \underline{M} , given by endomorphisms $u_i : M_i \rightarrow M_i$ ($i = 1, 2, 3$). Then we have $t_{M_2/A}(u_2) = t_{M_1/A}(u_1) + t_{M_3/A}(u_3)$.*

Proof. Suppose first that there exists a splitting $s : M_3 \rightarrow M_2$ for p , so that we can view u_2 as a matrix $\begin{pmatrix} u_1 & v \\ 0 & u_3 \end{pmatrix}$, where $v \in \text{Hom}_A(M_3, M_1)$. By additivity of the trace, we are then reduced to show that $t_{M_2/A}(i \circ v \circ p) = 0$. By lemma 3.2.5(i), this is the same as $t_{M_3/A}(p \circ i \circ v)$, which obviously vanishes. In general, for any $a \in \mathfrak{m}$ we consider the morphism $\mu_a = a \cdot \mathbf{1}_{M_3}$ and the pull back morphism $\underline{M} * \mu_a \rightarrow \underline{M}$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_1 & \xrightarrow{i} & M_2 & \xrightarrow{p} & M_3 \longrightarrow 0 \\ & & \parallel & & \uparrow \phi & & \uparrow \mu_a \\ 0 & \longrightarrow & M_1 & \longrightarrow & P & \longrightarrow & M_3 \longrightarrow 0 \end{array}$$

Then $\underline{M} * \mu_a$ is a split exact sequence with the endomorphism $\underline{u} * \mu_a = (u_1, v, u_3)$, for a certain $v \in \text{End}_A(P)$. The pair of morphisms $(a \cdot \mathbf{1}_{M_2}, p)$ induces a morphism $\psi : M_2 \rightarrow P$, and it is easy to check that $\phi \circ \psi = a \cdot \mathbf{1}_{M_2}$ and $\psi \circ \phi = a \cdot \mathbf{1}_P$. We can therefore apply lemma 3.2.5 to deduce that $a \cdot (t_{P/A}(v) - t_{M/A}(u)) = 0$. By the foregoing we know that $t_{P/A}(v) = t_{M_1/A}(u_1) + t_{M_3/A}(u_3)$, so the claim follows. \square

Suppose now that B is an almost finitely generated almost projective A -algebra. For any $b \in B_*$, denote by $\mu_b : B \rightarrow B$ the B -linear morphism $b' \mapsto b \cdot b'$. The map $b \mapsto \mu_b$ defines a B -linear monomorphism $\mu : B \rightarrow E(B)$. The composition

$$\text{Tr}_{B/A} = t_{B/A} \circ \mu : B \rightarrow A$$

will also be called the almost trace morphism of the A -algebra B .

Proposition 3.2.7. *Let A and B be as in the above discussion.*

- i) *If $\phi : A \rightarrow B$ is an isomorphism, then $\text{Tr}_{B/A} = \phi^{-1}$.*
- ii) *If C any other A -algebra, then $\text{Tr}_{C \otimes_A B/C} = \mathbf{1}_C \otimes_A \text{Tr}_{B/A}$.*
- iii) *If C is an almost projective almost finite B -algebra, then $\text{Tr}_{C/A} = \text{Tr}_{B/A} \circ \text{Tr}_{C/B}$.*

Proof. (i) and (ii) are left as exercises for the reader. We verify (iii). For given $\varepsilon, \delta \in \mathfrak{m}$ pick morphisms $B \xrightarrow{u_\varepsilon} A^n \xrightarrow{v_\varepsilon} B$ and $C \xrightarrow{u'_\delta} B^m \xrightarrow{v'_\delta} C$ such that $v_\varepsilon \circ u_\varepsilon = \varepsilon \cdot \mathbf{1}_B$ and $v'_\delta \circ u'_\delta = \delta \cdot \mathbf{1}_C$. If we set $u_\varepsilon^{\oplus m} = u_\varepsilon \otimes_A \mathbf{1}_{A^m}$, $u''_{\delta\varepsilon} = u_\varepsilon^{\oplus m} \circ u'_\delta : C \rightarrow A^n \otimes_A A^m$, $v_\varepsilon^{\oplus m} = v_\varepsilon \otimes_A \mathbf{1}_{A^m}$ and $v''_{\delta\varepsilon} = v'_\delta \circ v_\varepsilon^{\oplus m} : A^n \otimes_A A^m \rightarrow C$ then we have $v''_{\delta\varepsilon} \circ u''_{\delta\varepsilon} = \varepsilon \cdot \delta \cdot \mathbf{1}_C$. Define

$$\begin{array}{ll} t_{\varepsilon, B/A} : B \rightarrow A & b \mapsto \text{Tr}(u_\varepsilon \circ \mu_b \circ v_\varepsilon) \\ t_{\delta, C/B} : C \rightarrow B & c \mapsto \text{Tr}(u'_\delta \circ \mu_c \circ v'_\delta) \\ t_{\delta\varepsilon, C/A} : C \rightarrow A & c \mapsto \text{Tr}(u''_{\delta\varepsilon} \circ \mu_c \circ v''_{\delta\varepsilon}). \end{array}$$

Using (3.2.1) we can write

$$\begin{aligned} t_{\delta\varepsilon, C/A}(c) &= \sum_{\substack{i,j=1 \\ n,m}}^{n,m} (\pi_i^A \otimes_A \pi_j^A) \circ u''_{\delta\varepsilon} \circ \mu_c \circ v''_{\delta\varepsilon} \circ (e_i^A \otimes_A e_j^A) \\ &= \sum_{\substack{i,j=1 \\ n,m}}^{n,m} (\pi_i^A \otimes_A \pi_j^A) \circ u_\varepsilon^{\oplus m} \circ u'_\delta \circ \mu_c \circ v'_\delta \circ v_\varepsilon^{\oplus m} \circ (e_i^A \otimes_A e_j^A) \\ &= \sum_{\substack{i,j=1 \\ n}}^n \pi_i^A \circ u_\varepsilon \circ \pi_j^B \circ u'_\delta \circ \mu_c \circ v'_\delta \circ e_j^B \circ v_\varepsilon \circ e_i^A \\ &= \sum_{i=1}^n \pi_i^A \circ u_\varepsilon \circ t_{\delta, C/B}(c) \circ v_\varepsilon \circ e_i^A \\ &= t_{\varepsilon, B/A} \circ t_{\delta, C/B}(c) \end{aligned}$$

which implies immediately the claim. \square

Corollary 3.2.8. *Let $A \rightarrow B$ be a faithfully flat almost finitely presented and étale morphism of almost V -algebras. Then $\mathrm{Tr}_{B/A} : B \rightarrow A$ is an epimorphism.*

Proof. Under the stated hypotheses, B is an almost projective A -module (by proposition 2.3.15). Let $C = \mathrm{Coker}(\mathrm{Tr}_{B/A})$ and $\mathrm{Tr}_{B/B \otimes_A B}$ the trace morphism for the morphism of almost V -algebras $\mu_{B/A}$. By faithful flatness, the natural morphism $C \rightarrow C \otimes_A B = \mathrm{Coker}(\mathrm{Tr}_{B \otimes_A B/B})$ is a monomorphism, hence it suffices to show that $\mathrm{Tr}_{B \otimes_A B/B}$ is an epimorphism (here $B \otimes_A B$ is considered as a B -algebra via the second factor). However, from proposition 3.2.7(i) and (iii) we see that $\mathrm{Tr}_{B/B \otimes_A B}$ is a right inverse for $\mathrm{Tr}_{B \otimes_A B/B}$. The claim follows. \square

It is useful to introduce the A -linear morphism

$$\mathrm{tr}_{B/A} = \mathrm{Tr}_{B/A} \circ \mu_{B/A} : B \otimes_A B \rightarrow A.$$

We can view $\mathrm{tr}_{B/A}$ as a bilinear form; it induces an A -linear morphism

$$\tau_{B/A} : B \rightarrow B^* = \mathrm{alHom}_A(B, A)$$

characterized by the equality $\mathrm{tr}_{B/A}(b_1 \otimes b_2) = \tau_{B/A}(b_1)(b_2)$ for all $b_1, b_2 \in B_*$. We say that $\mathrm{tr}_{B/A}$ is a *perfect pairing* if $\tau_{B/A}$ is an isomorphism.

Theorem 3.2.9. *An almost projective and almost finite morphism $\phi : A \rightarrow B$ of almost V -algebras is étale if and only if the trace form $\mathrm{tr}_{B/A}$ is a perfect pairing.*

Proof. Suppose that ϕ is étale. Let $e_{B/A}$ be the idempotent almost element of $B \otimes_A B$ provided by proposition 3.1.4. We define a morphism $\sigma : B^* \rightarrow B$ by $f \mapsto (f \otimes_A \mathbf{1}_B)(e_{B/A})$. To start with, we remark that both $\tau_{B/A}$ and σ are B -linear morphisms (for the natural B -module structure of B^* defined in remark 2.3.18). Indeed, let us pick any $b, b', b'' \in B_*$, $f \in B^*$ and compute directly

$$\begin{aligned} (b \cdot \tau_{B/A}(b'))(b'') &= \tau_{B/A}(b')(bb'') = \mathrm{Tr}_{B/A}(bb'b'') = (\tau_{B/A}(bb'))(b''). \\ b \cdot \sigma(f) &= b \cdot (f \otimes_A \mathbf{1}_B)(e_{B/A}) = (f \otimes_A \mathbf{1}_B)((\mathbf{1}_{B/A} \otimes b) \cdot e_{B/A}) \\ &= (f \otimes_A \mathbf{1}_B)((b \otimes \mathbf{1}_{B/A}) \cdot e_{B/A}) = ((b \cdot f) \otimes_A \mathbf{1}_B)(e_{B/A}) \\ &= \sigma(b \cdot f). \end{aligned}$$

Next we show that σ is a left inverse for $\tau_{B/A}$. In fact, let $b \in B_*$. We have

$$\begin{aligned} \sigma \circ \tau_{B/A}(b) &= (\tau_{B/A}(b) \otimes_A \mathbf{1}_B)(e_{B/A}) = (\mathrm{Tr}_{B/A} \otimes_A \mathbf{1}_B)((b \otimes \mathbf{1}_{B/A}) \cdot e_{B/A}) \\ &= \mathrm{Tr}_{B \otimes_A B/B}((\mathbf{1}_{B/A} \otimes b) \cdot e_{B/A}) = b \cdot \mathrm{Tr}_{B \otimes_A B/B}(e_{B/A}). \end{aligned}$$

Therefore it suffices to show that $\mathrm{Tr}_{B \otimes_A B/B}(e_{B/A}) = \mathbf{1}$. However, by hypothesis ϕ is unramified, hence corollary 3.1.8 gives a decomposition $B \otimes_A B \simeq B \oplus I_{B/A}$ such that $e_{B/A}$ acts as the identity on the first factor and as the zero morphism on the second factor.

Now, let $X = \mathrm{Ker}(\sigma)$. From the above we derive a B -linear isomorphism $B^* \simeq B \oplus X$. We dualize and apply lemma 2.3.23(ii) to obtain another B -linear isomorphism

$$(3.2.10) \quad B \simeq (B^*)^* \simeq (B \oplus X)^* \simeq B^* \oplus X^* \simeq B \oplus X \oplus X^*.$$

Finally, composing the isomorphism (3.2.10) with the projection on the first factor, we get a split B -linear epimorphism $B \rightarrow B$, hence a *surjective* B_* -linear morphism $B_* \rightarrow B_*$. Such a morphism is necessarily an isomorphism, and, tracing backward, the same must hold for $\tau_{B/A}$.

Conversely, suppose that the trace form is a perfect pairing. By lemma 2.3.23(i) the natural morphism $\alpha : B^* \otimes_A B \rightarrow \mathrm{alHom}_B(B \otimes_A B, B)$ is an isomorphism and one verifies easily that $\alpha \circ (\tau_{B/A} \otimes_A \mathbf{1}_B) = \tau_{B \otimes_A B/B}$. In particular $\tau_{B \otimes_A B/B}$ is also an isomorphism. The multiplication gives an almost element $\mu_{B/A} \in \mathrm{alHom}_B(B \otimes_A B, B)_*$; let $e = \tau_{B \otimes_A B/B}^{-1}(\mu_{B/A})$. We derive

$$(3.2.11) \quad \mathrm{Tr}_{B \otimes_A B/B}(e) = \tau_{B \otimes_A B/B}(e)(\mathbf{1}_{B \otimes_A B}) = \mu_{B/A}(\mathbf{1}_{B \otimes_A B}) = \mathbf{1}_{B/A}.$$

Furthermore, we have already remarked that $\tau_{B/A}$ is a B -linear morphism, hence $\tau_{B \otimes_A B/B}$ is a $B \otimes_A B$ -linear morphism. Consequently, for any almost element x of $B \otimes_A B$ we have

$$\tau_{B \otimes_A B/B}(x \cdot e) = x \cdot \tau_{B \otimes_A B/B}(e) = x \cdot \mu_{B/A} = \mu_{B/A}(x) \cdot \mu_{B/A} = \mu_{B/A}(x) \cdot \tau_{B \otimes_A B/B}(e).$$

Since by hypothesis $\tau_{B/A}$ is an isomorphism, this implies

$$(3.2.12) \quad x \cdot e = \mu_{B/A}(x) \cdot e.$$

Consider the morphism $\mu_e : B \otimes_A B \rightarrow B \otimes_A B$ defined by $x \mapsto e \cdot x$; then μ_e is B -linear (for the B -module structure defined by the second factor). Applying (3.2.12) and lemma 3.2.4 we conclude that $t_{B \otimes_A B/B}(\mu_e) = \mu_{B/A}(e)$. On the other hand, (3.2.11) says that this trace is equal to $\underline{1}_{B/A}$, hence

$$(3.2.13) \quad \mu_{B/A}(e) = \underline{1}_{B/A}.$$

Let $\beta : B \rightarrow B \otimes_A B$ be defined as $b \mapsto b \cdot e$. From (3.2.12) we see that both β and $\mu_{B/A}$ are $B \otimes_A B$ -linear morphisms and from (3.2.13) we know moreover that $\mu_{B/A} \circ \beta = \mathbf{1}_B$. By lemma 2.3.12 we deduce that B is an almost projective $B \otimes_A B$ -module, *i.e.* ϕ is unramified, as claimed. \square

Definition 3.2.14. The *nilradical* of an almost algebra A is the ideal $\text{nil}(A) = \text{nil}(A_*)^a$ (where, for a ring R , we denote by $\text{nil}(R)$ the ideal of nilpotent elements in R). We say that A is *reduced* if $\text{nil}(A) \simeq 0$.

Notice that, if R is a V -algebra, then every nilpotent ideal in R^a is of the form I^a , where I is a nilpotent ideal in R (indeed, it is of the form I^a where I is an ideal, and $\mathfrak{m} \cdot I$ is seen to be nilpotent). It follows easily that $\text{nil}(A)$ is the colimit of the nilpotent ideals in A ; moreover $\text{nil}(R)^a = \text{nil}(R^a)$. Using this one sees that $A/\text{nil}(A)$ is reduced.

Proposition 3.2.15. *Let $A \rightarrow B$ be an étale almost finitely presented morphism of almost algebras. If A is reduced then B is reduced as well.*

Proof. Under the stated hypothesis, B is an almost projective A -module (by virtue of proposition 2.3.15(ii)). Hence, for given $\varepsilon \in \mathfrak{m}$, pick a sequence of morphisms $B \xrightarrow{u_\varepsilon} A^n \xrightarrow{v_\varepsilon} B$ such that $v_\varepsilon \circ u_\varepsilon = \varepsilon \cdot \mathbf{1}_B$; with the notation of (3.2.3), define $\nu_b : A^n \rightarrow A^n$ by $\nu_b = v_\varepsilon \circ \mu_b \circ u_\varepsilon$, so that $t_\varepsilon(b) = \text{Tr}(\nu_b)$. One verifies easily that $\nu_b^m = \varepsilon^{m-1} \cdot \nu_b$ for all integers $m > 0$.

Now, suppose that $b \in \text{nil}(B_*)$. It follows that $b^m = 0$ for m sufficiently large, hence $\nu_b^m = 0$ for m sufficiently large. Let \mathfrak{p} be any prime ideal of A_* ; let $\pi : A_* \rightarrow A_*/\mathfrak{p}$ be the natural projection and F the fraction field of A_*/\mathfrak{p} . The F -linear morphism $\nu_{b*} \otimes_{A_*} \mathbf{1}_F$ is nilpotent on the vector space F^n , hence $\pi \circ \text{Tr}(\nu_{b*}) = \text{Tr}(\nu_{b*} \otimes_{A_*} \mathbf{1}_F) = 0$. This shows that $\text{Tr}(\nu_{b*})$ lies in the intersection of all prime ideals of A_* , hence it is nilpotent. Since by hypothesis A is reduced, we get $\text{Tr}(\nu_{b*}) = 0$. Finally, this implies $\text{Tr}_{B/A}(b) = 0$. Now, for any $b' \in B_*$, the almost element bb' will be nilpotent as well, so the same conclusion applies to it. This shows that $\tau_{B/A}(b) = 0$. But by hypothesis B is étale over A , hence theorem 3.2.9 yields $b = 0$, as required. \square

Remark 3.2.16. Let M be an A -module. We say that an almost element a of A is M -regular if the multiplication morphism $m \mapsto am : M \rightarrow M$ is a monomorphism. Assume **(A)** (cf. section 2.1) and suppose furthermore that \mathfrak{m} is generated by a multiplicative system \mathcal{S} which is a cofiltered semigroup under the preorder structure (\mathcal{S}, \succ) induced by the divisibility relation in V . We say that \mathcal{S} is archimedean if, for all $s, t \in \mathcal{S}$ there exists $n > 0$ such that $s^n \succ t$. Suppose that \mathcal{S} is archimedean and that A is a reduced almost algebra. Then \mathcal{S} consists of A -regular elements. Indeed, by hypothesis $\text{nil}(A_*)^a = 0$; since the annihilator of \mathcal{S} in A_* is 0 we get $\text{nil}(A_*) = 0$. Suppose that $s \cdot a = 0$ for some $s \in \mathcal{S}$ and $a \in A_*$. Let $t \in \mathcal{S}$ be arbitrary and pick $n > 0$ such that $t^n \succ s$. Then $(ta)^n = 0$ hence $ta = 0$ for all $t \in \mathcal{S}$, hence $a = 0$.

3.3. Lifting theorems. Throughout the following, the terminology “epimorphism of V^a -algebras” will refer to a morphism of V^a -algebras that induces an epimorphism on the underlying V^a -modules.

Lemma 3.3.1. *Let $A \rightarrow B$ be an epimorphism of almost V -algebras with kernel I . Let U be the A -extension $0 \rightarrow I/I^2 \rightarrow A/I^2 \rightarrow B \rightarrow 0$. Then the assignment $f \mapsto f * U$ defines a natural isomorphism*

$$(3.3.2) \quad \text{Hom}_B(I/I^2, M) \xrightarrow{\sim} \text{Exal}_A(B, M).$$

Proof. Let $X = (0 \rightarrow M \rightarrow E \xrightarrow{p} B \rightarrow 0)$ be any A -extension of B by M . The composition $g : A \rightarrow E \xrightarrow{p} B$ of the structural morphism for E followed by p coincides with the projection $A \rightarrow B$. Therefore $g(I) \subset M$ and $g(I^2) = 0$. Hence g factors through A/I^2 ; the restriction of g to I/I^2 defines a morphism $f \in \text{Hom}_B(I/I^2, M)$ and a morphism of A -extensions $f * U \rightarrow X$. In this way we obtain an inverse for (3.3.2). \square

Now consider any morphism of A -extensions

$$(3.3.3) \quad \begin{array}{ccccccc} \tilde{B} : & 0 & \longrightarrow & I & \longrightarrow & B & \longrightarrow B_0 \longrightarrow 0 \\ & \downarrow \tilde{f} & & \downarrow u & & \downarrow f & & \downarrow f_0 \\ \tilde{C} : & 0 & \longrightarrow & J & \longrightarrow & C & \longrightarrow C_0 \longrightarrow 0. \end{array}$$

The morphism u induces by adjunction a morphism of C_0 -modules

$$(3.3.4) \quad C_0 \otimes_{B_0} I \rightarrow J$$

whose image is the ideal $I \cdot C$, so that the square diagram of almost algebras defined by \tilde{f} is cofibred (i.e. $C_0 \simeq C \otimes_B B_0$) if and only if (3.3.4) is an epimorphism.

Lemma 3.3.5. *Let $\tilde{f} : \tilde{B} \rightarrow \tilde{C}$ be a morphism of A -extensions as above, such that the corresponding square diagram of almost algebras is cofibred. Then the morphism $f : B \rightarrow C$ is flat if and only if $f_0 : B_0 \rightarrow C_0$ is flat and (3.3.4) is an isomorphism.*

Proof. It follows directly from the (almost version of the) local flatness criterion (see [18] Th. 22.3). \square

We are now ready to put together all the work done so far and begin the study of deformations of almost algebras.

The morphism $u : I \rightarrow J$ is an element in $\text{Hom}_{B_0}(I, J)$; by lemma 3.3.1 the latter group is naturally isomorphic to $\text{Exal}_B(B_0, J)$. By applying transitivity (theorem 2.4.18) to the sequence of morphisms $B \rightarrow B_0 \xrightarrow{f_0} C_0$ we obtain an exact sequence of abelian groups

$$\text{Exal}_{B_0}(C_0, J) \rightarrow \text{Exal}_B(C_0, J) \rightarrow \text{Hom}_{B_0}(I, J) \xrightarrow{\partial} \text{Ext}_{C_0!!}^2(\mathbb{L}_{C_0/B_0}, J_!).$$

Hence we can form the element $\omega(\tilde{B}, f_0, u) = \partial(u) \in \text{Ext}_{C_0!!}^2(\mathbb{L}_{C_0/B_0}, J_!)$. The proof of the next result goes exactly as in [14] (III.2.1.2.3).

Proposition 3.3.6. *i) Let the A -extension \tilde{B} , the B_0 -linear morphism $u : I \rightarrow J$ and the morphism of A -algebras $f_0 : B_0 \rightarrow C_0$ be given as above. Then there exists an A -extension \tilde{C} and a morphism $\tilde{f} : \tilde{B} \rightarrow \tilde{C}$ completing diagram (3.3.3) if and only if $\omega(\tilde{B}, f_0, u) = 0$. (i.e. $\omega(\tilde{B}, f_0, u)$ is the obstruction to the lifting of \tilde{B} over f_0 .)*

ii) Assume that the obstruction $\omega(\tilde{B}, f_0, u)$ vanishes. Then the set of isomorphism classes of A -extensions \tilde{C} as in (i) forms a torsor under the group $\text{Exal}_{B_0}(C_0, J) (\simeq \text{Ext}_{C_0!!}^1(\mathbb{L}_{C_0/B_0}, J_!))$.

iii) The group of automorphisms of an A -extension \tilde{C} as in (i) is naturally isomorphic to $\text{Der}_{B_0}(C_0, J) (\simeq \text{Ext}_{C_0!!}^0(\mathbb{L}_{C_0/B_0}, J_!))$. \square

The obstruction $\omega(\tilde{B}, f_0, u)$ depends functorially on u . More exactly, if we denote by

$$\omega(\tilde{B}, f_0) \in \mathbb{E}xt_{C_{0!!}}^2(\mathbb{L}_{C_0/B_0}, (C_0 \otimes_{B_0} I)_!)$$

the obstruction corresponding to the natural morphism $I \rightarrow C_0 \otimes_{B_0} I$, then for any other morphism $u : I \rightarrow J$ we have

$$\omega(\tilde{B}, f_0, u) = v_! \circ \omega(\tilde{B}, f_0)$$

where v is the morphism (3.3.4). Taking lemma 3.3.5 into account we deduce

Corollary 3.3.7. *Suppose that $B_0 \rightarrow C_0$ is flat. Then*

i) *The class $\omega(\tilde{B}, f_0)$ is the obstruction to the existence of a flat deformation of C_0 over B , i.e. of a B -extension \tilde{C} as in (3.3.3) such that C is flat over B and $C \otimes_B B_0 \rightarrow C_0$ is an isomorphism.*

ii) *If the obstruction $\omega(\tilde{B}, f_0)$ vanishes, then the set of isomorphism classes of flat deformations of C_0 over B forms a torsor under the group $\text{Exal}_{B_0}(C_0, C_0 \otimes_{B_0} I)$.*

iii) *The group of automorphisms of a given flat deformation of C_0 over B is naturally isomorphic to $\text{Der}_{B_0}(C_0, C_0 \otimes_{B_0} I)$. \square*

Now, suppose we are given two A -extensions \tilde{C}^1, \tilde{C}^2 with morphisms of A -extensions

$$\begin{array}{ccccccc} \tilde{B} : & 0 & \longrightarrow & I & \longrightarrow & B & \longrightarrow B_0 \longrightarrow 0 \\ & & & \downarrow u^i & & \downarrow f^i & \downarrow f_0^i \\ \tilde{C}^i : & 0 & \longrightarrow & J^i & \longrightarrow & C^i & \longrightarrow C_0^i \longrightarrow 0 \end{array}$$

and morphisms $v : J^1 \rightarrow J^2, g_0 : C_0^1 \rightarrow C_0^2$ such that

$$(3.3.8) \quad u^2 = v \circ u^1 \quad \text{and} \quad f_0^2 = g_0 \circ f_0^1.$$

We consider the problem of finding a morphism of A -extensions

$$(3.3.9) \quad \begin{array}{ccccccc} \tilde{C}^1 : & 0 & \longrightarrow & J^1 & \longrightarrow & C^1 & \longrightarrow C_0^1 \longrightarrow 0 \\ & & & \downarrow v & & \downarrow g & \downarrow g_0 \\ \tilde{C}^2 : & 0 & \longrightarrow & J^2 & \longrightarrow & C^2 & \longrightarrow C_0^2 \longrightarrow 0 \end{array}$$

such that $\tilde{f}^2 = \tilde{g} \circ \tilde{f}^1$. Let us denote by $e(\tilde{C}^i) \in \mathbb{E}xt_{C_{0!!}}^1(\mathbb{L}_{C_0^i/B}, J_!^i)$ the classes defined by the B -extensions \tilde{C}^1, \tilde{C}^2 via the isomorphism of theorem 2.4.17 and by

$$\begin{aligned} v_* : \mathbb{E}xt_{C_{0!!}}^1(\mathbb{L}_{C_0^1/B}, J_!^1) &\rightarrow \mathbb{E}xt_{C_{0!!}}^1(\mathbb{L}_{C_0^1/B}, J_!^2) \\ *g_0 : \mathbb{E}xt_{C_{0!!}}^1(\mathbb{L}_{C_0^2/B}, J_!^2) &\rightarrow \mathbb{E}xt_{C_{0!!}}^1(C_{0!!}^2 \otimes_{C_{0!!}} \mathbb{L}_{C_0^1/B}, J_!^2) \end{aligned}$$

the canonical morphisms defined by v and g_0 . Using the natural isomorphism

$$\mathbb{E}xt_{C_{0!!}}^1(\mathbb{L}_{C_0^1/B}, J_!^2) \simeq \mathbb{E}xt_{C_{0!!}}^1(C_{0!!}^2 \otimes_{C_{0!!}} \mathbb{L}_{C_0^1/B}, J_!^2)$$

we can identify the target of both v_* and $*g_0$ with $\mathbb{E}xt_{C_{0!!}}^1(\mathbb{L}_{C_0^1/B}, J_!^2)$. It is clear that the problem admits a solution if and only if the A -extensions $v_* \tilde{C}^1$ and $\tilde{C}^2 * g_0$ coincide, i.e. if and only if $v_* e(\tilde{C}^1) - e(\tilde{C}^2) * g_0 = 0$. By applying transitivity to the sequence of morphisms $B \rightarrow B_0 \rightarrow C_0^1$ we obtain an exact sequence

$$\mathbb{E}xt_{C_{0!!}}^1(\mathbb{L}_{C_0^1/B_0}, J_!^2) \hookrightarrow \mathbb{E}xt_{C_{0!!}}^1(\mathbb{L}_{C_0^1/B}, J_!^2) \rightarrow \text{Hom}_{C_0^1}(C_0^1 \otimes_{B_0} I, J^2)$$

It follows from (3.3.8) that the image of $v * e(\tilde{C}^1) - e(\tilde{C}^2) * g_0$ in the group $\text{Hom}_{C_0^1}(C_0^1 \otimes_{B_0} I, J^2)$ vanishes, therefore

$$(3.3.10) \quad v * e(\tilde{C}^1) - e(\tilde{C}^2) * g_0 \in \mathbb{E}xt_{C_0^{!!}}^1(\mathbb{L}_{C_0^1/B_0}, J_!^2).$$

In conclusion, we derive the following result as in [14] (III.2.2.2).

Proposition 3.3.11. *With the above notations, the class (3.3.10) is the obstruction to the existence of a morphism of A -extensions $\tilde{g} : \tilde{C}^1 \rightarrow \tilde{C}^2$ as in (3.3.9) such that $\tilde{f}^2 = \tilde{g} \circ \tilde{f}^1$. When the obstruction vanishes, the set of such morphisms forms a torsor under the group $\text{Der}_{B_0}(C_0^1, J^2)$ (the latter being identified with $\mathbb{E}xt_{C_0^{!!}}^0(C_{0!!}^2 \otimes_{C_0^{!!}} \mathbb{L}_{C_0^1/B_0}, J_!^2)$). \square*

For a given almost V -algebra A , we define the category $\mathbf{w}\text{-}\acute{\text{E}}\mathbf{t}(A)$ as the full subcategory of $A\text{-}\mathbf{Alg}$ consisting of all weakly étale A -algebras. Notice that, by lemma 3.1.2(iv) all morphisms in $\mathbf{w}\text{-}\acute{\text{E}}\mathbf{t}(A)$ are weakly étale.

Theorem 3.3.12. *i) Let $A \rightarrow B$ be a weakly étale morphism of almost algebras. Let C be any A -algebra and $I \subset C$ a nilpotent ideal. Then the natural morphism*

$$\text{Hom}_{A\text{-}\mathbf{Alg}}(B, C) \rightarrow \text{Hom}_{A\text{-}\mathbf{Alg}}(B, C/I)$$

is bijective.

ii) Let A be a V^a -algebra, $I \subset A$ a nilpotent ideal and $A' = A/I$. Then the natural functor

$$\mathbf{w}\text{-}\acute{\text{E}}\mathbf{t}(A) \rightarrow \mathbf{w}\text{-}\acute{\text{E}}\mathbf{t}(A') \quad (\phi : A \rightarrow B) \mapsto (\mathbf{1}_{A'} \otimes_A \phi : A' \rightarrow A' \otimes_A B)$$

is an equivalence of categories.

iii) The equivalence of (ii) restricts to an equivalence $\acute{\text{E}}\mathbf{t}(A) \rightarrow \acute{\text{E}}\mathbf{t}(A')$.

Proof. By induction we can assume $I^2 = 0$. Then (i) follows directly from proposition 3.3.11 and theorem 2.4.23. We show (ii) : by corollary 3.3.7 (and again theorem 2.4.23) a given weakly étale morphism $\phi' : A' \rightarrow B'$ can be lifted to a *unique* flat morphism $\phi : A \rightarrow B$. We need to prove that ϕ is weakly étale, i.e. that B is $B \otimes_A B$ -flat. However, it is clear that $\mu_{B'/A'} : B' \otimes_{A'} B' \rightarrow B'$ is weakly étale, hence it has a flat lifting $\tilde{\mu} : B \otimes_A B \rightarrow C$. Then the composition $A \rightarrow B \otimes_A B \rightarrow C$ is flat and it is a lifting of ϕ' . We deduce that there is an isomorphism of A -algebras $\alpha : B \rightarrow C$ lifting $\mathbf{1}_{B'}$ and moreover the morphisms $b \mapsto \tilde{\mu}(b \otimes \mathbf{1})$ and $b \mapsto \tilde{\mu}(\mathbf{1} \otimes b)$ coincide with α . Claim (ii) follows. To show (iii), suppose that $A' \rightarrow B'$ is étale and let $I_{B'/A'}$ denote as usual the kernel of $\mu_{B'/A'}$. By corollary 3.1.8 there is a natural morphism of almost algebras $B' \otimes_{A'} B' \rightarrow I_{B'/A'}$ which is clearly étale. Hence $I_{B'/A'}$ lifts to a weakly étale $B \otimes_A B$ -algebra C , and the isomorphism $B' \otimes_{A'} B' \simeq I_{B'/A'} \oplus B'$ lifts to an isomorphism $B \otimes_A B \simeq C \oplus B$ of $B \otimes_A B$ -algebras. It follows that B is an almost projective $B \otimes_A B$ -module, i.e. $A \rightarrow B$ is étale, as claimed. \square

We conclude with some results on deformations of almost modules. These can be established independently of the theory of the cotangent complex, along the lines of [14] (IV.3.1.12).

We begin by recalling some notation from *loc. cit.* Let R be a ring and $J \subset R$ an ideal with $J^2 = 0$. Set $R' = R/J$; an extension of R -modules $\underline{M} = (0 \rightarrow K \rightarrow M \xrightarrow{p} M' \rightarrow 0)$ where K and M' are killed by J , defines a natural morphism of R' -modules $u(\underline{M}) : J \otimes_{R'} M' \rightarrow K$ such that $u(\underline{M})(x \otimes m') = xm$ for $x \in J$, $m \in M$ and $p(m) = m'$. By the local flatness criterion ([18] Th.22.3) M is flat over R if and only if M' is flat over R' and $u(\underline{M})$ is an isomorphism. One can then show the following.

Proposition 3.3.13. (cp. [14] (IV.3.1.5))

i) Given R' -modules M' and K and a morphism $u' : J \otimes_{R'} M' \rightarrow K$ there exists an obstruction $\omega(R, u') \in \text{Ext}_{R'}^2(M', K)$ whose vanishing is necessary and sufficient for the existence of an extension of R -modules \underline{M} of M' by K such that $u(\underline{M}) = u'$.

ii) When $\omega(R, u') = 0$, the set of isomorphism classes of such extensions \underline{M} forms a torsor under $\text{Ext}_{R'}^1(M', K)$; the group of automorphisms of such an extension is $\text{Hom}_{R'}(M', K)$. \square

Lemma 3.3.14. *Let $A \rightarrow B$ be a finite morphism of almost algebras with nilpotent kernel. Let $\phi : M \rightarrow N$ be an A -linear morphism and set $\phi_B = \phi \otimes_A \mathbf{1}_B : M \otimes_A B \rightarrow N \otimes_A B$. Then there exists $m \geq 0$ such that*

i) $\text{Ann}_A(\text{Coker}(\phi_B))^m \subset \text{Ann}_A(\text{Coker}(\phi))$.

ii) $(\text{Ann}_V(\text{Ker}(\phi_B)) \cdot \text{Ann}_V(\text{Tor}_1^A(B, N)) \cdot \text{Ann}_V(\text{Coker}(\phi)))^m \subset \text{Ann}_A(\text{Ker}(\phi))$.

If $B = A/I$ for some nilpotent ideal I , and $I^n = 0$, then we can take $m = n$ in (i) and (ii).

Proof. Under the assumptions, we can find a finitely generated A_* -module Q such that $\mathfrak{m} \cdot B_* \subset Q \subset B_*$. By [12] (1.1.5), there exists a finite filtration $0 = J_m \subset \dots \subset J_1 \subset J_0 = A_*$ such that each J_i/J_{i+1} is a quotient of a direct sum of copies of Q . This implies that, for every A -module M , we have

$$(3.3.15) \quad \text{Ann}_A(M \otimes_A B)^m \subset \text{Ann}_A(M).$$

(i) follows easily. Notice that if $B = A/I$ and $I^n = 0$, then we can take $m = n$ in (3.3.15).

For (ii) let $C^\bullet = \text{Cone}(\phi)$. We estimate $H = H^{-1}(C^\bullet \otimes_A B)$ in two ways. By the first spectral sequence of hyperhomology we have an exact sequence $\text{Tor}_1^A(N, B) \rightarrow H \rightarrow \text{Ker}(\phi_B)$. By the second spectral sequence for hyperhomology we have an exact sequence $\text{Tor}_2^A(\text{Coker}(\phi), B) \rightarrow \text{Ker}(\phi) \otimes_A B \rightarrow H$. Hence $\text{Ker}(\phi) \otimes_A B$ is annihilated by the product of the three annihilators in (ii) and the result follows by applying (3.3.15) with $M = \text{Ker}(\phi)$. \square

Lemma 3.3.16. *Keep the assumptions of lemma 3.3.14, let M be an A -module and set $M_B = B \otimes_A M$.*

i) *If $A \rightarrow B$ is an epimorphism, M is flat and M_B is almost projective over B , then M is almost projective over A .*

ii) *If M_B is an almost finitely generated B -module then M is an almost finitely generated A -module.*

iii) *If $\text{Tor}_1^A(B, M) = 0$ and M_B is almost finitely presented over B , then M is almost finitely presented over A .*

Proof. (i) : we have to show that $\text{Ext}_A^1(M, N)$ is almost zero for every A -module N . Let $I = \text{Ker}(A \rightarrow B)$; by assumption I is nilpotent, so by the usual devissage we may assume that $I \cdot N = 0$. If $\chi \in \text{Ext}_A^1(M, N)$ is represented by an extension $0 \rightarrow N \rightarrow Q \rightarrow M \rightarrow 0$ then after tensoring by B and using the flatness of M we get an exact sequence of B -modules $0 \rightarrow N \rightarrow B \otimes_A Q \rightarrow M_B \rightarrow 0$. Thus χ comes from an element of $\text{Ext}_B^1(M_B, N)$ which is almost zero by assumption.

(ii) : let $\mathfrak{m}_0 = (\varepsilon_1, \dots, \varepsilon_m)$ be a finitely generated subideal of \mathfrak{m} . By assumption there is a map $\phi' : B^r \rightarrow M_B$ such that $\mathfrak{m}_0 \cdot \text{Coker}(\phi') = 0$. For all $j \leq m$ the morphism $\varepsilon_j \cdot \phi'$ lifts to a morphism $\phi_j : A^r \rightarrow M$. Then $\phi = \phi_1 \oplus \dots \oplus \phi_m : A^{rm} \rightarrow M$ satisfies $\mathfrak{m}_0^2 \cdot \text{Coker}(\phi \otimes_A \mathbf{1}_B) = 0$. By lemma 3.3.14(i) it follows $\mathfrak{m}_0^{2n} \cdot \text{Coker}(\phi) = 0$ for some $n \geq 0$. As \mathfrak{m}_0 was arbitrary, the result follows.

(iii) Let \mathfrak{m}_0 be as above. By (ii), M is almost finitely generated over A , so we can choose a morphism $\phi : A^r \rightarrow M$ such that $\mathfrak{m}_0 \cdot \text{Coker}(\phi) = 0$. Consider $\phi_B = \phi \otimes_A \mathbf{1}_B : B^r \rightarrow M_B$. By lemma 2.3.6, there is a finitely generated submodule N of $\text{Ker}(\phi_B)$ containing $\mathfrak{m}_0^2 \cdot \text{Ker}(\phi_B)$. Notice that $\text{Ker}(\phi) \otimes_A B$ maps onto $\text{Ker}(B^r \rightarrow \text{Im}(\phi) \otimes_A B)$ and $\text{Ker}(\text{Im}(\phi) \otimes_A B \rightarrow M_B) \simeq \text{Tor}_1^A(B, \text{Coker}(\phi))$ is annihilated by \mathfrak{m}_0 . Hence $\mathfrak{m}_0 \cdot \text{Ker}(\phi_B)$ is contained in the image of $\text{Ker}(\phi)$ and therefore we can lift a finite generating set $\{x'_1, \dots, x'_n\}$ for $\mathfrak{m}_0^2 \cdot N$ to almost elements $\{x_1, \dots, x_n\}$ of $\text{Ker}(\phi)$. If we quotient A^r by the span of these x_i , we get a finitely presented A -module F with a morphism $\bar{\phi} : F \rightarrow M$ such that $\text{Ker}(\bar{\phi} \otimes_A B)$ is annihilated by \mathfrak{m}_0^4 and

$\text{Coker}(\bar{\phi})$ is annihilated by \mathfrak{m}_0 . By lemma 3.3.14(ii) we derive $\mathfrak{m}_0^{5m} \cdot \text{Ker}(\bar{\phi}) = 0$ for some $m \geq 0$. Since \mathfrak{m}_0 is arbitrary, this proves the result. \square

Remark 3.3.17. (i) Inspecting the proof, one sees that parts (ii) and (iii) of lemma 3.3.16 hold whenever (3.3.15) holds. For instance, if $A \rightarrow B$ is any faithfully flat morphism, then (3.3.15) holds with $m = 1$.

ii) Consequently, if $A \rightarrow B$ is faithfully flat and M is an A -module such that M_B is flat (resp. almost finitely generated, resp. almost finitely presented) over B , then M is flat (resp. almost finitely generated, resp. almost finitely presented) over A .

iii) On the other hand, we do not know whether a general faithfully flat morphism $A \rightarrow B$ descends almost projectivity. However, using (ii) and proposition 2.3.15 we see that if the B -module M_B is almost finitely generated almost projective, then M has the same property.

iv) However, if B is faithfully flat and almost finitely presented as an A -module, then $A \rightarrow B$ does descend almost projectivity, as can be easily deduced from lemma 2.3.23(i) and proposition 2.3.15(ii).

Theorem 3.3.18. *Let I be a nilpotent ideal of the almost algebra A and set $A' = A/I$. Suppose that $\tilde{\mathfrak{m}}$ is a (flat) V -module of homological dimension ≤ 1 . Let P' be an almost projective A' -module.*

i) *There is an almost projective A -module P with $A' \otimes_A P \simeq P'$.*

ii) *If P' is almost finitely presented, then P is almost finitely presented.*

Proof. As usual we reduce to $I^2 = 0$. Then proposition 3.3.13(i) applies with $R = A_*$, $J = I_*$, $R' = A_*/I_*$, $M' = P'_!$, $K = I_* \otimes_{R'} P'_!$ and $u' = \mathbf{1}_K$. We obtain a class $\omega(A_*, u') \in \text{Ext}_{R'}^2(P'_!, I_* \otimes_{R'} P'_!)$ which gives the obstruction to the existence of a flat A_* -module F lifting $P'_!$. Since $P'_!$ is almost projective, we know that $\mathfrak{m} \cdot \text{Ext}_{R'}^2(P'_!, I_* \otimes_{R'} P'_!) = 0$, which says that $0 = \varepsilon \cdot \omega(A_*, u') = \omega(A_*, \varepsilon \cdot u')$ for all $\varepsilon \in \mathfrak{m}$. In other words, for every $\varepsilon \in \mathfrak{m}$ we can find an extension of A_* -modules $\underline{P}_\varepsilon$ of $P'_!$ by $I_* \otimes_{R'} P'_!$ such that $u(\underline{P}_\varepsilon) = \varepsilon \cdot \mathbf{1}_{I_* \otimes_{R'} P'_!}$. Let $\chi_\varepsilon \in \text{Ext}_{A_*}^1(P'_!, I_* \otimes_{R'} P'_!)$ be the class of $\underline{P}_\varepsilon$. Notice that, for any $\delta \in \mathfrak{m}$, $\delta \cdot \chi_\varepsilon$ is the class of an extension \underline{X} such that $u(\underline{X}) = \delta \cdot u(\underline{P}_\varepsilon) = \delta \cdot \varepsilon \cdot \mathbf{1}_{I_* \otimes_{R'} P'_!}$, hence, by proposition 3.3.13(ii), $\gamma \cdot (\delta \cdot \chi_\varepsilon - \chi_{\delta \cdot \varepsilon}) = 0$ for all $\gamma \in \mathfrak{m}$. Hence we can define a morphism

$$\chi : \mathfrak{m} \otimes_V \mathfrak{m} \otimes_V \mathfrak{m} \rightarrow \text{Ext}_{A_*}^1(P'_!, I_* \otimes_{R'} P'_!) \quad \varepsilon \otimes \delta \otimes \gamma \mapsto \delta \cdot \gamma \cdot \chi_\varepsilon.$$

However, one sees easily that $\mathfrak{m} \otimes_V \mathfrak{m} \otimes_V \mathfrak{m} \simeq \tilde{\mathfrak{m}}$ and $\tilde{\mathfrak{m}} \otimes_V P'_! \simeq P'_!$, hence we can view χ as an element of $\text{Hom}_V(\tilde{\mathfrak{m}}, \text{Ext}_{A_*}^1(P'_!, I_* \otimes_{R'} P'_!))$ and moreover we have a spectral sequence

$$E_2^{pq} = \text{Ext}_V^p(\tilde{\mathfrak{m}}, \text{Ext}_{A_*}^q(P'_!, I_* \otimes_{R'} P'_!)) \Rightarrow \text{Ext}_{A_*}^{p+q}(P'_!, I_* \otimes_{R'} P'_!)$$

with $E_2^{pq} = 0$ for all $p \geq 2$ (this spectral sequence is constructed e.g. from the double complex $\text{Hom}_V(F_p, \text{Hom}_{A_*}(F'_q, I_* \otimes_{R'} P'_!))$ where F_\bullet (resp. F'_\bullet) is a projective resolution of $\tilde{\mathfrak{m}}$ (resp. $P'_!$). In particular, our χ is an element in E_2^{01} which therefore survives in the abutment as a class of E_∞^{01} . The latter can be lifted to an element $\tilde{\chi}$ via the surjection $\text{Ext}_{A_*}^1(P'_!, I_* \otimes_{R'} P'_!) \rightarrow E_\infty^{01}$. Let $0 \rightarrow I_* \otimes_{R'} P'_! \rightarrow Q \rightarrow P'_! \rightarrow 0$ be an extension representing $\tilde{\chi}$. Checking compatibilities, we see that $\delta \cdot \varepsilon \cdot \tilde{\chi} = \delta \cdot \chi_\varepsilon$ for every $\varepsilon, \delta \in \mathfrak{m}$. Hence $u(\tilde{\chi}) : I_* \otimes_{R'} P'_! \rightarrow I_* \otimes_{R'} P'_!$ coincides with the identity map on the submodule $\mathfrak{m} \cdot I_* \otimes_{R'} P'_!$. Since $\mathfrak{m} \cdot P'_! = P'_!$, we see that $u(\tilde{\chi})$ is actually the identity map. By the local flatness criterion, it then follows that Q is flat over R , hence the A -module $P = Q^a$ is a flat lifting of P' , so it is almost projective, by lemma 3.3.16(i). Now (ii) follows from (i), lemma 3.3.16(ii) and proposition 2.3.15(i). \square

Remark 3.3.19. (i) According to proposition 2.1.10(ii), theorem 3.3.18 applies especially when \mathfrak{m} is countably generated as a V -module.

(ii) For P and P' as in theorem 3.3.18(ii) let $\sigma_P : P \rightarrow P'$ be the projection. It is natural to ask whether the pair (P, σ_P) is uniquely determined up to isomorphism, i.e. whether, for any other

pair $(Q, \sigma_Q : Q \rightarrow P')$ for which theorem 3.3.18 holds, there exists an A -linear isomorphism $\phi : P \rightarrow Q$ such that $\sigma_Q \circ \phi = \sigma_P$. The answer is negative in general. Consider the case $P' = A'$. Take $P = Q = A$ and let σ_P be the natural projection, while $\sigma_Q = (u' \cdot \mathbf{1}_{A'}) \circ \sigma_P$, where u' is a unit in A'_* . Then the uniqueness question amounts to whether every unit in A'_* lifts to a unit of A_* . The following counterexample is related to the fact that the completion of the algebraic closure $\overline{\mathbb{Q}_p}$ of \mathbb{Q}_p is not maximally complete. Let $V = \overline{\mathbb{Z}_p}$, the integral closure of \mathbb{Z}_p in $\overline{\mathbb{Q}_p}$. Then V is a non-discrete valuation ring of rank one, and we take \mathfrak{m} as in example 2.1.1(i), $A = (V/p^2V)^a$ and $A' = A/pA$. Choose a compatible system of roots of p . An almost element of A' is just a V -linear morphism $\phi : \operatorname{colim}_{n>0} p^{1/n!}V \rightarrow V/pV$. Such a ϕ can be represented (in a non-unique way) by an infinite series of the form $\sum_{n=1}^{\infty} a_n p^{1-1/n!}$ ($a_n \in V$). The meaning of this expression is as follows. For every $m > 0$, scalar multiplication by the element $\sum_{n=1}^m a_n p^{1-1/n!} \in V$ defines a morphism $\phi_m : p^{1/m!}V \rightarrow V/pV$. For $m' > m$, let $j_{m,m'} : p^{1/m!}V \rightarrow p^{1/m'!}V$ be the imbedding. Then we have $\phi_{m'} \circ j_{m,m'} = \phi_m$, so that we can define $\phi = \operatorname{colim}_{m>0} \phi_m$. Similarly, every almost element of A can be represented by an expression of the form $a_0 + \sum_{n=1}^{\infty} a_n p^{2-1/n!}$. Now, if $\sigma : A \rightarrow A'$ is the natural projection, the induced map $\sigma_* : A_* \rightarrow A'_*$ is given by: $a_0 + \sum_{n=1}^{\infty} a_n p^{2-1/n!} \mapsto a_0$. In particular, its image is the subring $V/p \subset (V/p)_* = A'_*$. For instance, the unit $\sum_{n=1}^{\infty} p^{1-1/n!}$ of A'_* does not lie in the image of this map.

In the light of the above remark, the best one can achieve in general is the following result.

Proposition 3.3.20. *Assume (A) (see section 2.1) and keep the notation of theorem 3.3.18. Suppose moreover that $(Q, \sigma_Q : Q \rightarrow P')$ and $(P, \sigma_P : P \rightarrow P')$ are two pairs as in remark 3.3.19. Then for all $\varepsilon \in \mathfrak{m}$ there exist A -linear morphisms $t_\varepsilon : P \rightarrow Q$ and $s_\varepsilon : Q \rightarrow P$ such that*

$$\begin{aligned} \mathbf{PQ}(\varepsilon) \quad & \sigma_Q \circ t_\varepsilon = \varepsilon \cdot \sigma_P & \sigma_P \circ s_\varepsilon = \varepsilon \cdot \sigma_Q \\ & s_\varepsilon \circ t_\varepsilon = \varepsilon^2 \cdot \mathbf{1}_P & t_\varepsilon \circ s_\varepsilon = \varepsilon^2 \cdot \mathbf{1}_Q. \end{aligned}$$

Proof. Since both Q and P are almost projective and σ_P, σ_Q are epimorphisms, there exist morphisms $\bar{t}_\varepsilon : P \rightarrow Q$ and $\bar{s}_\varepsilon : Q \rightarrow P$ such that $\sigma_Q \circ \bar{t}_\varepsilon = \varepsilon \cdot \sigma_P$ and $\sigma_P \circ \bar{s}_\varepsilon = \varepsilon \cdot \sigma_Q$. Then we have $\sigma_P \circ (\bar{s}_\varepsilon \circ \bar{t}_\varepsilon - \varepsilon^2 \cdot \mathbf{1}_P) = 0$ and $\sigma_Q \circ (\bar{t}_\varepsilon \circ \bar{s}_\varepsilon - \varepsilon^2 \cdot \mathbf{1}_Q) = 0$, i.e. the morphism $u_\varepsilon = \varepsilon^2 \cdot \mathbf{1}_P - \bar{s}_\varepsilon \circ \bar{t}_\varepsilon$ (resp. $v_\varepsilon = \varepsilon^2 \cdot \mathbf{1}_Q - \bar{t}_\varepsilon \circ \bar{s}_\varepsilon$) has image contained in the almost submodule IP (resp. IQ). Since $I^m = 0$ this implies $u_\varepsilon^m = 0$ and $v_\varepsilon^m = 0$. Hence

$$\varepsilon^{2m} \cdot \mathbf{1}_P = (\varepsilon^2 \mathbf{1}_P)^m - u_\varepsilon^m = \left(\sum_{a=0}^{m-1} \varepsilon^{2a} u_\varepsilon^{m-1-a} \right) \circ \bar{s}_\varepsilon \circ \bar{t}_\varepsilon.$$

Define $\bar{s}_{(2m-1)\varepsilon} = \left(\sum_{a=0}^{m-1} \varepsilon^{2a} u_\varepsilon^{m-1-a} \right) \circ \bar{s}_\varepsilon$. Notice that $\bar{s}_{(2m-1)\varepsilon} = \bar{s}_\varepsilon \circ \left(\sum_{a=0}^{m-1} \varepsilon^{2a} v_\varepsilon^{m-1-a} \right)$. This implies the equalities $\bar{s}_{(2m-1)\varepsilon} \circ \bar{t}_\varepsilon = \varepsilon^{2m} \cdot \mathbf{1}_P$ and $\bar{t}_\varepsilon \circ \bar{s}_{(2m-1)\varepsilon} = \varepsilon^{2m} \cdot \mathbf{1}_Q$. Then the pair $(\bar{s}_{(2m-1)\varepsilon}, \varepsilon^{2(m-1)} \cdot \bar{t}_\varepsilon)$ satisfies $\mathbf{PQ}(\varepsilon^{2m-1})$. Under (A), every element of \mathfrak{m} is a multiple of an element of the form ε^{2m-1} , therefore the claim follows for arbitrary $\varepsilon \in \mathfrak{m}$. \square

3.4. Descent. Faithfully flat descent in the almost setting presents no particular surprises: since the functor $A \mapsto A_\#$ preserves faithful flatness of morphisms (see remark 3.1.3) many well-known results for usual rings and modules extend *verbatim* to almost algebras. So for instance, faithfully flat morphisms are of universal effective descent for the fibred categories $F : V^a\text{-Alg}.\text{Mod}^o \rightarrow V^a\text{-Alg}^o$ and $G : V^a\text{-Alg}.\text{Morph}^o \rightarrow V^a\text{-Alg}^o$ (see definition 2.4.12: for an almost V -algebra B , the fibre F_B (resp. G_B) is the opposite of the category of B -modules (resp. B -algebras)). Then, using remark 3.3.17, we deduce also universal effective descent for the fibred subcategories of flat (resp. almost finitely generated, resp. almost finitely presented, resp. almost projective almost finitely generated) modules. Likewise, a faithfully flat morphism

is of universal effective descent for the fibred subcategories $\dot{\mathbf{E}t}^o \rightarrow V^a\text{-}\mathbf{Alg}^o$ of étale (resp. $\mathbf{w}\dot{\mathbf{E}t}^o \rightarrow V^a\text{-}\mathbf{Alg}^o$ of weakly étale) algebras.

More generally, since the functor $A \mapsto A_{!!}$ preserves pure morphisms in the sense of [20], and since, by a theorem of Olivier (*loc. cit.*), pure morphisms are of universal effective descent for modules, the same holds for pure morphisms of almost algebras.

Non-flat descent is more delicate. Our results are not as complete here as it could be wished, but nevertheless, they suffice for current applications (namely, for the cases needed in [6]).

Our first statement is the almost version of a theorem of Gruson and Raynaud (cp. [13] (Part II, Th.1.2.4)).

Proposition 3.4.1. *A finite monomorphism of almost algebras descends flatness.*

Proof. Let $\phi : A \rightarrow B$ be such a morphism. Under the assumption, we can find a finite A_* -module Q such that $\mathfrak{m} \cdot B_* \subset Q \subset B_*$. One sees easily that Q is a faithful A_* -module, so by [13] (Part II, Th.1.2.4 and lemma 1.2.2), Q satisfies the following condition :

(3.4.2) If $(0 \rightarrow N \rightarrow L \rightarrow P \rightarrow 0)$ is an exact sequence of A_* -modules with L flat, such that $\text{Im}(N \otimes_{A_*} Q)$ is a pure submodule of $L \otimes_{A_*} Q$, then P is flat.

Now let M be an A -module such that $M \otimes_A B$ is flat. Pick an epimorphism $p : F \rightarrow M$ with F free over A . Then $\underline{Y} = (0 \rightarrow \text{Ker}(p \otimes_A \mathbf{1}_B) \rightarrow F \otimes_A B \rightarrow M \otimes_A B \rightarrow 0)$ is universally exact over B , hence over A . Consider the sequence $\underline{X} = (0 \rightarrow \text{Im}(\text{Ker}(p)_! \otimes_{A_*} Q) \rightarrow F_! \otimes_{A_*} Q \rightarrow M_! \otimes_{A_*} Q \rightarrow 0)$. Clearly $\underline{X}^a \simeq \underline{Y}$. However, it is easy to check that a sequence \underline{E} of A -modules is universally exact if and only if the sequence $\underline{E}_!$ is universally exact over A_* . We conclude that $\underline{X} = (\underline{X}^a)_!$ is a universally exact sequence of A_* -modules, hence, by condition (3.4.2), $M_!$ is flat over A_* , i.e. M is flat over A as required. \square

Corollary 3.4.3. *Let $A \rightarrow B$ be a finite morphism of almost algebras, with nilpotent kernel. If C is a flat A -algebra such that $C \otimes_A B$ is weakly étale (resp. étale) over B , then C is weakly étale (resp. étale) over A .*

Proof. In the weakly étale case, we have to show that the multiplication morphism $\mu : C \otimes_A C \rightarrow C$ is flat. As $N = \text{Ker}(A \rightarrow B)$ is nilpotent, the local flatness criterion reduces the question to the situation over A/N . So we may assume that $A \rightarrow B$ is a monomorphism. Then $C \otimes_A C \rightarrow (C \otimes_A C) \otimes_A B$ is a monomorphism, but $\mu \otimes_{C \otimes_A C} \mathbf{1}_{(C \otimes_A C) \otimes_A B}$ is the multiplication morphism of $C \otimes_A B$, which is flat by assumption. Therefore, by proposition 3.4.1, μ is flat.

For the étale case, we have to show that C is almost finitely presented as a $C \otimes_A C$ -module. By hypothesis $C \otimes_A B$ is almost finitely presented as a $C \otimes_A C \otimes_A B$ -module and we know already that C is flat as a $C \otimes_A C$ -module, so by lemma 3.3.16(iii) (applied to the finite morphism $C \otimes_A C \rightarrow C \otimes_A C \otimes_A B$) the claim follows. \square

Next we consider the following situation. We are given a cartesian diagram of almost algebras

$$(3.4.4) \quad \begin{array}{ccc} A_0 & \xrightarrow{f_2} & A_2 \\ f_1 \downarrow & & \downarrow g_2 \\ A_1 & \xrightarrow{g_1} & A_3 \end{array}$$

such that one of the morphisms $A_i \rightarrow A_3$ ($i = 1, 2$) is an epimorphism. We denote by \mathcal{M}_i (resp. $\mathcal{M}_{i,\text{fl}}$, resp. $\mathcal{M}_{i,\text{proj}}$) the category of all (resp. flat, resp. almost projective) A_i -modules, for $i = 0, \dots, 3$. Diagram (3.4.4) induces an essentially commutative diagram for the corresponding categories \mathcal{M}_i , where the arrows are given by the “extension of scalars” functors. There follows a natural functor

$$\pi : \mathcal{M}_0 \rightarrow \mathcal{M}_1 \times_{\mathcal{M}_3} \mathcal{M}_2$$

from \mathcal{M}_0 to the 2-fibred products of \mathcal{M}_1 and \mathcal{M}_2 over \mathcal{M}_3 . Recall (cp. [1] (Ch.VII §3)) that $\mathcal{M}_1 \times_{\mathcal{M}_3} \mathcal{M}_2$ is the category whose objects are the triples (M_1, M_2, ξ) , where M_i is an A_i -module ($i = 1, 2$) and $\xi : A_3 \otimes_{A_1} M_1 \xrightarrow{\sim} A_3 \otimes_{A_2} M_2$ is an A_3 -linear isomorphism. Given such an object (M_1, M_2, ξ) , let us denote $M_3 = A_3 \otimes_{A_2} M_2$; we have a natural morphism $M_2 \rightarrow M_3$, and ξ gives a morphism $M_1 \rightarrow M_3$, so we can form the fibre product $T(M_1, M_2, \xi) = M_1 \times_{M_3} M_2$. In this way we obtain a functor $T : \mathcal{M}_1 \times_{\mathcal{M}_3} \mathcal{M}_2 \rightarrow \mathcal{M}_0$, and we leave to the reader the verification that T is right adjoint to π . Let us denote by $\varepsilon : \mathbf{1}_{\mathcal{M}_0} \rightarrow T \circ \pi$ and $\eta : \pi \circ T \rightarrow \mathbf{1}_{\mathcal{M}_1 \times_{\mathcal{M}_3} \mathcal{M}_2}$ the unit and counit of the adjunction.

Lemma 3.4.5. *The functor π induces an equivalence of full subcategories :*

$$\{X \in \text{Ob}(\mathcal{M}_0) \mid \varepsilon_X \text{ is an isomorphism}\} \xrightarrow{\pi} \{Y \in \text{Ob}(\mathcal{M}_1 \times_{\mathcal{M}_3} \mathcal{M}_2) \mid \eta_Y \text{ is an isomorphism}\}$$

having T as essential inverse.

Proof. General nonsense. □

Lemma 3.4.6. *Let M be any A_0 -module. Then ε_M is an epimorphism. If M is flat over A_0 , ε_M is an isomorphism.*

Proof. Indeed, $\varepsilon_M : M \rightarrow (A_1 \otimes_{A_0} M) \times_{A_3 \otimes_{A_0} M} (A_2 \otimes_{A_0} M)$ is the natural morphism. So, the assertions follow by applying $- \otimes_{A_0} M$ to the short exact sequence of A_0 -modules

$$(3.4.7) \quad 0 \rightarrow A_0 \xrightarrow{f} A_1 \oplus A_2 \xrightarrow{g} A_3 \rightarrow 0$$

where $f(a) = (f_1(a), f_2(a))$ and $g(a, b) = g_1(a) - g_2(b)$. □

There is another case of interest, in which ε_M is an isomorphism. Namely, suppose that one of the morphisms $A_i \rightarrow A_3$ ($i = 1, 2$), say $A_1 \rightarrow A_3$, has a section. Then also the morphism $A_0 \rightarrow A_2$ gains a section $s : A_2 \rightarrow A_0$ and we have the following :

Lemma 3.4.8. *In the above situation, suppose that the A_0 -module M arises by extension of scalars from an A_2 -module M' , via the section $s : A_2 \rightarrow A_0$. Then ε_M is an isomorphism.*

Proof. Indeed, in this case, (3.4.7) is split exact as a sequence of A_2 -modules, and it remains such after tensoring by M' . □

Lemma 3.4.9. *$\eta_{(M_1, M_2, \xi)}$ is an isomorphism for all objects (M_1, M_2, ξ) .*

Proof. To fix ideas, suppose that $A_1 \rightarrow A_3$ is an epimorphism. Consider any object (M_1, M_2, ξ) of $\mathcal{M}_1 \times_{\mathcal{M}_3} \mathcal{M}_2$. Let $M = T(M_1, M_2, \xi)$; we deduce a natural morphism

$$\phi : (M \otimes_{A_0} A_1) \times_{M \otimes_{A_0} A_3} (M \otimes_{A_0} A_2) \rightarrow M_1 \times_{M_3} M_2$$

such that $\phi \circ \varepsilon_M = \mathbf{1}_M$. It follows that ε_M is injective, hence it is an isomorphism, by lemma 3.4.6. We derive a commutative diagram with exact rows :

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & (M \otimes_{A_0} A_1) \oplus (M \otimes_{A_0} A_2) & \longrightarrow & M \otimes_{A_0} A_3 \longrightarrow 0 \\ & & \parallel & & \downarrow \phi_1 \oplus \phi_2 & & \downarrow \phi_3 \\ 0 & \longrightarrow & M & \longrightarrow & M_1 \oplus M_2 & \longrightarrow & M_3 \longrightarrow 0. \end{array}$$

From the snake lemma we deduce

$$\begin{aligned} (*) & \quad \text{Ker}(\phi_1) \oplus \text{Ker}(\phi_2) \simeq \text{Ker}(\phi_3) \\ (**) & \quad \text{Coker}(\phi_1) \oplus \text{Coker}(\phi_2) \simeq \text{Coker}(\phi_3). \end{aligned}$$

Since $M_3 \simeq M_1 \otimes_{A_1} A_3$ we have $A_3 \otimes_{A_1} \text{Coker}(\phi_1) \simeq \text{Coker}(\phi_3)$. But by assumption $A_1 \rightarrow A_3$ is an epimorphism, so also $\text{Coker}(\phi_1) \rightarrow \text{Coker}(\phi_3)$ is an epimorphism. Then $(**)$ implies that $\text{Coker}(\phi_2) = 0$. But $\phi_3 = \mathbf{1}_{A_3} \otimes_{A_2} \phi_2$, thus $\text{Coker}(\phi_3) = 0$ as well. We look at the

exact sequence $0 \rightarrow \text{Ker}(\phi_1) \rightarrow M \otimes_{A_0} A_1 \xrightarrow{\phi_1} M_1 \rightarrow 0$: applying $A_3 \otimes_{A_1} -$ we obtain an epimorphism $A_3 \otimes_{A_1} \text{Ker}(\phi_1) \rightarrow \text{Ker}(\phi_3)$. From (*) it follows that $\text{Ker}(\phi_2) = 0$. In conclusion, ϕ_2 is an isomorphism. Hence the same is true for $\phi_3 = \mathbf{1}_{A_3} \otimes_{A_2} \phi_2$, and again (*), (**) show that ϕ_1 is an isomorphism as well, which implies the claim. \square

Lemma 3.4.10. *If $(A_1 \times A_2) \otimes_{A_0} M$ is flat over $A_1 \times A_2$, then M is flat over A_0 .*

Proof. Suppose that $A_1 \rightarrow A_3$ is an epimorphism and let I be its kernel. Let $\tilde{A} = A_{1!!} \times_{A_{3!!}} A_{2!!}$; it suffices to show that M_I is a flat \tilde{A} -module. However, in view of proposition 2.3.27, the assumption implies that $(A_{1!!} \times A_{2!!}) \otimes_{\tilde{A}} M_I$ is a flat $A_{1!!} \times A_{2!!}$ -module. I_I is the kernel of the epimorphism $A_{1!!} \rightarrow A_{3!!}$. Moreover, I_I identifies naturally with an ideal of \tilde{A} and $\tilde{A}/I_I \simeq A_{2!!}$. Then the desired conclusion follows from [7] (lemma in *loc. cit.*). \square

Proposition 3.4.11. *The functor π restricts to equivalences :*

$$\begin{aligned} \mathcal{M}_{0,\text{fl}} &\xrightarrow{\sim} \mathcal{M}_{1,\text{fl}} \times_{\mathcal{M}_{3,\text{fl}}} \mathcal{M}_{2,\text{fl}} \\ \mathcal{M}_{0,\text{proj}} &\xrightarrow{\sim} \mathcal{M}_{1,\text{proj}} \times_{\mathcal{M}_{3,\text{proj}}} \mathcal{M}_{2,\text{proj}}. \end{aligned}$$

Proof. The assertion for flat almost modules follows directly from lemmata 3.4.5, 3.4.6, 3.4.9 and 3.4.10. Set $B = A_1 \times A_2$. To establish the second equivalence, it suffices to show that, if P is an A_0 -module such that $B \otimes_{A_0} P$ is almost projective over B , then P is almost projective over A_0 , or which is the same, that $\text{alExt}_{A_0}^i(P, N) \simeq 0$ for all $i > 0$ and any A_0 -module N . We know already that P is flat. Let M be any A_0 -module and N any B -module. The standard isomorphism $R\text{Hom}_B(B \overset{\mathbf{L}}{\otimes}_{A_0} M, N) \simeq R\text{Hom}_{A_0}(M, N)$ yields a natural isomorphism $\text{alExt}_B^i(B \otimes_{A_0} M, N) \simeq \text{alExt}_{A_0}^i(M, N)$, whenever $\text{Tor}_j^{A_0}(B, M) = 0$ for every $j > 0$. In particular, we have $\text{alExt}_{A_0}^i(P, N) \simeq 0$ whenever N comes from either an A_1 -module, or an A_2 -module. For a general A_0 -module N there is a 3-step filtration such that $\text{Fil}_0(N) = 0$, $\text{gr}_1(N) = \text{Fil}_1(N) = \text{Ker}(\varepsilon_N)$, $\text{gr}_2(N) = \text{Ker}(A_1 \otimes_{A_0} N \rightarrow A_3 \otimes_{A_0} N)$ and $\text{gr}_3(N) = A_2 \otimes_{A_0} N$. By an easy devissage, we reduce to verify that $\text{alExt}_{A_0}^i(P, \text{gr}_j(N)) = 0$ for every $i > 0$ and $j = 1, 2, 3$. However, $\text{gr}_2(N)$ is an A_1 -module and $\text{gr}_3(N)$ is an A_2 -module, so the required vanishing follows for $j = 2, 3$. Moreover, applying $- \otimes_{A_0} N$ to (3.4.7), we derive a short exact sequence :

$$(3.4.12) \quad 0 \rightarrow \text{Tor}_1^{A_0}(N, A_2) \rightarrow \frac{\text{Tor}_1^{A_0}(N, A_3)}{\text{Tor}_1^{A_0}(N, A_1)} \rightarrow \text{gr}_1(N) \rightarrow 0.$$

Here again, the leftmost term of (3.4.12) is an A_2 -module, and the middle term is an A_1 -module, so the same devissage yields the sought vanishing also for $j = 1$. \square

Corollary 3.4.13. *In the situation of (3.4.4), denote by $\mathcal{A}_{i,\text{fl}}$ (resp. $\hat{\mathbf{E}}\mathbf{t}_i$, resp. $\mathbf{w}.\hat{\mathbf{E}}\mathbf{t}_i$) the category of flat (resp. étale, resp. weakly étale) A_i -algebras. The functor π induces equivalences*

$$\mathcal{A}_{0,\text{fl}} \xrightarrow{\sim} \mathcal{A}_{1,\text{fl}} \times_{\mathcal{A}_{3,\text{fl}}} \mathcal{A}_{2,\text{fl}} \quad \hat{\mathbf{E}}\mathbf{t}_0 \xrightarrow{\sim} \hat{\mathbf{E}}\mathbf{t}_1 \times_{\hat{\mathbf{E}}\mathbf{t}_3} \hat{\mathbf{E}}\mathbf{t}_2 \quad \mathbf{w}.\hat{\mathbf{E}}\mathbf{t}_0 \xrightarrow{\sim} \mathbf{w}.\hat{\mathbf{E}}\mathbf{t}_1 \times_{\mathbf{w}.\hat{\mathbf{E}}\mathbf{t}_3} \mathbf{w}.\hat{\mathbf{E}}\mathbf{t}_2. \quad \square$$

Next we want to reinterpret the equivalences of proposition 3.4.11 in terms of descent data. If $F : \mathcal{C} \rightarrow V^a\text{-Alg}^o$ is a fibred category over the opposite of the category of almost algebras, and if $X \rightarrow Y$ is a given morphism of almost algebras, we shall denote by $\text{Desc}(\mathcal{C}, Y/X)$ the category of objects of the fibre category F_Y , endowed with a descent datum relative to the morphism $X \rightarrow Y$ (cp. [9] (Ch.II §1)). In the arguments hereafter, we consider morphisms of almost algebras and modules, and one has to reverse the direction of the arrows to pass to morphisms in the considered fibred category. Denote by $p_i : Y \rightarrow Y \otimes_X Y$ ($i = 1, 2$), resp. $p_{ij} : Y \otimes_X Y \rightarrow Y \otimes_X Y \otimes_X Y$ ($1 \leq i < j \leq 3$) the usual morphisms. As an example, $\text{Desc}(V^a\text{-Alg}^o, Y/X)$ consists of the pairs (M, β) where M is a Y -module and β is a

$Y \otimes_X Y$ -linear isomorphism $\beta : p_2^*(M) \xrightarrow{\sim} p_1^*(M)$ such that

$$(3.4.14) \quad p_{12}^*(\beta) \circ p_{23}^*(\beta) = p_{13}^*(\beta).$$

Let now $I \subset X$ be an ideal, and set $\overline{X} = X/I$, $\overline{Y} = Y/I \cdot Y$. For any $F : \mathcal{C} \rightarrow V^a\text{-Alg}^o$ as above, one has an essentially commutative diagram:

$$\begin{array}{ccc} \text{Desc}(\mathcal{C}, Y/X) & \longrightarrow & \text{Desc}(\mathcal{C}, \overline{Y}/\overline{X}) \\ \downarrow & & \downarrow \\ F_Y & \longrightarrow & F_{\overline{Y}}. \end{array}$$

This induces a functor :

$$(3.4.15) \quad \text{Desc}(\mathcal{C}, Y/X) \rightarrow \text{Desc}(\mathcal{C}, \overline{Y}/\overline{X}) \times_{F_{\overline{Y}}} F_Y.$$

Lemma 3.4.16. *With the above notation, suppose moreover that the natural morphism $I \rightarrow I \cdot Y$ is an isomorphism. Then the functor (3.4.15) is an equivalence whenever \mathcal{C} is one of the fibred categories $V^a\text{-Alg}\cdot\text{Mod}^o$, $V^a\text{-Alg}\cdot\text{Morph}^o$, $\hat{\text{Et}}^o$, $\text{w}\cdot\hat{\text{Et}}^o$.*

Proof. For any $n > 0$, denote by $Y^{\otimes n}$ (resp. $\overline{Y}^{\otimes n}$) the n -fold tensor product of Y (resp. \overline{Y}) with itself over X (resp. \overline{X}), and by $\rho_n : Y^{\otimes n} \rightarrow \overline{Y}^{\otimes n}$ the natural morphism. First of all we claim that, for every $n > 0$, the natural diagram of almost algebras

$$(3.4.17) \quad \begin{array}{ccc} Y^{\otimes n} & \xrightarrow{\rho_n} & \overline{Y}^{\otimes n} \\ \mu_n \downarrow & & \downarrow \overline{\mu}_n \\ Y & \xrightarrow{\rho_1} & \overline{Y} \end{array}$$

is cartesian (where μ_n and $\overline{\mu}_n$ are n -fold multiplication morphisms). For this, we need to verify that, for every $n > 0$, the induced morphism $\text{Ker}(\rho_n) \rightarrow \text{Ker}(\rho_1)$ (defined by multiplication of the first two factors) is an isomorphism. It then suffices to check that the natural morphism $\text{Ker}(\rho_n) \rightarrow \text{Ker}(\rho_{n-1})$ is an isomorphism for all $n > 1$. Indeed, consider the commutative diagram

$$\begin{array}{ccccccc} I \otimes_X Y^{\otimes n-1} & \xrightarrow{p} & I \cdot Y^{\otimes n-1} & \xrightarrow{i} & Y^{\otimes n-1} & & \\ \parallel & & \psi \downarrow & & \phi \otimes \mathbf{1}_{Y^{\otimes n-1}} \downarrow & \searrow \mathbf{1}_{Y^{\otimes n-1}} & \\ I \otimes_X Y^{\otimes n-1} & \xrightarrow{p'} & \text{Ker}(\rho_n) & \xrightarrow{i'} & Y^{\otimes n} & \xrightarrow{\mu_{Y/X} \otimes \mathbf{1}_{Y^{\otimes n-2}}} & Y^{\otimes n-1} \end{array}$$

From $I \cdot Y = \phi(Y)$, it follows that p' is an epimorphism. Hence also ψ is an epimorphism. Since i is a monomorphism, it follows that ψ is also a monomorphism, hence ψ is an isomorphism and the claim follows easily.

We consider first the case $\mathcal{C} = V^a\text{-Alg}\cdot\text{Mod}^o$; we see that (3.4.17) is a diagram of the kind considered in (3.4.4), hence, for every $n > 0$, we have the associated functor $\pi_n : Y^{\otimes n}\text{-Mod} \rightarrow \overline{Y}^{\otimes n}\text{-Mod} \times_{\overline{Y}\text{-Mod}} Y\text{-Mod}$ and also its right adjoint T_n . Denote by $\overline{p}_i : \overline{Y} \rightarrow \overline{Y}^{\otimes 2}$ ($i = 1, 2$) the usual morphisms, and similarly define $\overline{p}_{ij} : \overline{Y}^{\otimes 2} \rightarrow \overline{Y}^{\otimes 3}$. Suppose there is given a descent datum $(\overline{M}, \overline{\beta})$ for \overline{M} , relative to $\overline{X} \rightarrow \overline{Y}$. The cocycle condition (3.4.14) implies easily that $\overline{\mu}_2^*(\overline{\beta})$ is the identity on $\overline{\mu}_2^*(\overline{p}_i^*(\overline{M})) = \overline{M}$. It follows that the pair $(\overline{\beta}, \mathbf{1}_M)$ defines an isomorphism $\pi_2(p_1^*M) \xrightarrow{\sim} \pi_2(p_2^*M)$ in the category $\overline{Y}^{\otimes 2}\text{-Mod} \times_{\overline{Y}\text{-Mod}} Y\text{-Mod}$. Hence $T_2(\overline{\beta}, \mathbf{1}_M) : T_2 \circ \pi_2(p_1^*M) \rightarrow T_2 \circ \pi_2(p_2^*M)$ is an isomorphism. However, we remark that either morphism \overline{p}_i yields a section for μ_2 , hence we are in the situation contemplated in lemma 3.4.8, and we derive an isomorphism $\beta : p_2^*(M) \xrightarrow{\sim} p_1^*(M)$. We claim that (M, β) is an object of $\text{Desc}(\mathcal{C}, Y/X)$, i.e. that β verifies the cocycle condition (3.4.14). Indeed, we can

compute: $\pi_3(p_{ij}^*\beta) = (\rho_3^*(p_{ij}^*\beta), \mu_3^*(p_{ij}^*\beta))$ and by construction we have $\rho_3^*(p_{ij}^*\beta) = \bar{p}_{ij}^*(\bar{\beta})$ and $\mu_3^*(p_{ij}^*\beta) = \mu_2^*(\beta) = \mathbf{1}_M$. Therefore, the cocycle identity for $\bar{\beta}$ implies the equality $\pi_3(p_{12}^*(\bar{\beta})) \circ \pi_3(p_{23}^*(\bar{\beta})) = \pi_3(p_{13}^*(\bar{\beta}))$. If we now apply the functor T_3 to this equality, and then invoke again lemma 3.4.8, the required cocycle identity for β will ensue. Clearly β is the only descent datum on M lifting $\bar{\beta}$. This proves that (3.4.15) is essentially surjective. The same sort of argument also shows that the functor (3.4.15) induces bijections on morphisms, so the lemma follows in this case. Next, the case $\mathcal{C} = V^a\text{-Alg.Morph}^o$ can be deduced formally from the previous case, by applying repeatedly natural isomorphisms of the kind $p_i^*(M \otimes_Y N) \simeq p_i^*(M) \otimes_{Y \otimes_X Y} p_i^*(N)$ ($i = 1, 2$). Finally, the “étaleness” of an object of $\text{Desc}(V^a\text{-Alg.Morph}^o, Y/X)$ can be checked on its projection onto $Y\text{-Alg}^o$, hence also the cases $\mathcal{C} = \mathbf{w}\text{-}\acute{\text{E}}\mathbf{t}^o$ and $\mathcal{C} = \acute{\text{E}}\mathbf{t}^o$ follow directly. \square

Now, let $B = A_1 \times A_2$; to an objet (M, β) in $\text{Desc}(V^a\text{-Alg.Mod}^o, B/A)$ we assign an object (M_1, M_2, ξ) of $\mathcal{M}_1 \times_{\mathcal{M}_3} \mathcal{M}_2$, as follows. Set $M_i = A_i \otimes_B M$ ($i = 1, 2$) and $A_{ij} = A_i \otimes_{A_0} A_j$. We can write $B \otimes_{A_0} B = \prod_{i,j=1}^2 A_{ij}$ and β gives rise to the A_{ij} -linear isomorphisms $\beta_{ij} : A_{ij} \otimes_{B \otimes_{A_0} B} p_2^*(M) \xrightarrow{\sim} A_{ij} \otimes_{B \otimes_{A_0} B} p_1^*(M)$. In other words, we obtain isomorphisms $\beta_{ij} : A_i \otimes_{A_0} M_j \rightarrow M_i \otimes_{A_0} A_j$. However, we have a natural isomorphism $A_{12} \simeq A_3$ (indeed, suppose that $A_1 \rightarrow A_3$ is an epimorphism with kernel I ; then I is also an ideal of A_0 and $A_0/I \simeq A_2$; now the claim follows by remarking that $I \cdot A_1 = I$). Hence we can choose $\xi = \beta_{12}$. In this way we obtain a functor :

$$(3.4.18) \quad \text{Desc}(V^a\text{-Alg.Mod}^o, B/A_0) \rightarrow (\mathcal{M}_1 \times_{\mathcal{M}_3} \mathcal{M}_2)^o.$$

Proposition 3.4.19. *The functor (3.4.18) is an equivalence of categories.*

Proof. Let us say that $A_1 \rightarrow A_3$ is an epimorphism with kernel I . Then I is also an ideal of B and we have $B/I \simeq A_3 \times A_2$ and $A_0/I \simeq A_2$. We intend to apply lemma 3.4.16 to the morphism $A_0 \rightarrow B$. However, the induced morphism $\bar{B} = B/I \rightarrow \bar{A}_0 = A_0/I$ in $V^a\text{-Alg}^o$ has a section, and hence it is of universal effective descent for every fibred category. Thus, we can replace in (3.4.15) the category $\text{Desc}(V^a\text{-Alg.Mod}^o, \bar{B}/\bar{A}_0)$ by $\bar{A}_0\text{-Mod}^o$, and thereby, identify (up to equivalence) the target of (3.4.15) with the 2-fibred product $(\mathcal{M}_1 \times_{\mathcal{M}_2} \mathcal{M}_2)^o \times_{(\mathcal{M}_3 \times \mathcal{M}_2)^o} \mathcal{M}_2^o$. The latter is equivalent to the category $\mathcal{M}_1^o \times_{\mathcal{M}_3} \mathcal{M}_2^o$ and the resulting functor $\text{Desc}(V^a\text{-Alg.Mod}^o, B/A_0) \rightarrow \mathcal{M}_1^o \times_{\mathcal{M}_3} \mathcal{M}_2^o$ is canonically isomorphic to (3.4.18), which gives the claim. \square

Putting together propositions 3.4.11 and 3.4.19 we obtain the following :

Corollary 3.4.20. *In the situation of (3.4.4), the morphism $A_0 \rightarrow A_1 \times A_2$ is of effective descent for the fibred categories of flat almost modules and of almost projective almost modules.* \square

Next we would like to give sufficient conditions to ensure that a morphism of almost algebras is of effective descent for the fibred category $\mathbf{w}\text{-}\acute{\text{E}}\mathbf{t}^o \rightarrow V^a\text{-Alg}^o$ of weakly étale algebras (resp. for étale algebras). To this aim we are led to the following :

Definition 3.4.21. A morphism $\phi : A \rightarrow B$ of almost algebras is said to be *strictly finite* if $\text{Ker}(\phi)$ is nilpotent and $B \simeq R^a$, where R is a finite A_* -algebra.

Theorem 3.4.22. *Let $\phi : A \rightarrow B$ be a strictly finite morphism of almost algebras. Then :*

- i) *For every A -algebra C , the induced morphism $C \rightarrow C \otimes_A B$ is again strictly finite.*
- ii) *If M is a flat A -module and $B \otimes_A M$ is almost projective over B , then M is almost projective over A .*
- iii) *$A \rightarrow B$ is of universal effective descent for the fibred categories of weakly étale (resp. étale) almost algebras.*

Proof. (i): suppose that $B = R^a$ for a finite A_* -algebra R ; then $S = C_* \otimes_{A_*} R$ is a finite C_* -algebra and $S^a \simeq C$. It remains to show that $\text{Ker}(C \rightarrow C \otimes_A B)$ is nilpotent. Suppose that R is generated by n elements as an A_* -module and let $F_{A_*}(R)$ (resp. $F_{C_*}(S)$) be the Fitting ideal of R (resp. of S); we have $\text{Ann}_{C_*}(S)^n \subset F_{C_*}(S) \subset \text{Ann}_{C_*}(S)$ (see [15] (Chap.XIX Prop.2.5)); on the other hand $F_{C_*}(S) = F_{A_*}(R) \cdot C_*$, so the claim is clear.

(iii): we shall consider the fibred category $F : \mathbf{w.Et}^o \rightarrow V^a\text{-Alg}^o$; the same argument applies also to étale almost algebras. We begin by establishing a very special case :

Claim 3.4.23. Assertion (iii) holds when $B = (A/I_1) \times (A/I_2)$, where I_1 and I_2 are ideals in A such that $I_1 \cap I_2$ is nilpotent.

Proof of the claim: First of all we remark that the situation considered in the claim is stable under arbitrary base change, therefore it suffices to show that ϕ is of F -2-descent in this case. Then we factor ϕ as a composition $A \rightarrow A/\text{Ker}(\phi) \rightarrow B$ and we remark that $A \rightarrow A/\text{Ker}(\phi)$ is of F -2-descent by theorem 3.3.12; since a composition of morphisms of F -2-descent is again of F -2-descent, we are reduced to show that $A/\text{Ker}(\phi) \rightarrow B$ is of F -2-descent, i.e. we can assume that $\text{Ker}(\phi) \simeq 0$. However, in this case the claim follows easily from corollary 3.4.20.

Claim 3.4.24. More generally, assertion (iii) holds when $B = \prod_{i=1}^n A/I_i$, where I_1, \dots, I_n are ideals of A , such that $\bigcap_{i=1}^n I_i$ is nilpotent.

Proof of the claim: We prove this by induction on n , the case $n = 2$ being covered by claim 3.4.23. Therefore, suppose that $n > 2$, and set $B' = A/(\bigcap_{i=1}^{n-1} I_i)$. By induction, the morphism $B' \rightarrow \prod_{i=1}^{n-1} A/I_i$ is of universal F -2-descent. However, according to [9] (Chap.II Prop.1.1.3), the sieves of universal F -2-descent form a topology on $V^a\text{-Alg}^o$; for this topology, $\{A, B\}$ is a covering family of $A \times B$ and $(A \rightarrow B' \times (A/I_n))^o$ is a covering morphism, hence $\{B', A/I_n\}$ is a covering family of A , and then, by composition of covering families, $\{\prod_{i=1}^{n-1} A/I_i, A/I_n\}$ is a covering family of A , which is equivalent to the claim.

Now, let $A \rightarrow B$ be a general strictly finite morphism, so that $B = R^a$ for some finite A_* -algebra R . Pick generators f_1, \dots, f_m of the A_* -module R , and monic polynomials $p_1(X), \dots, p_m(X)$ such that $p_i(f_i) = 0$ for $i = 1, \dots, m$.

Claim 3.4.25. There exists a finite and faithfully flat extension C of A_* such that the images in $C[X]$ of $p_1(X), \dots, p_m(X)$ split as products of monic linear factors.

Proof of the claim: This extension C can be obtained as follows. It suffices to find, for each $i = 1, \dots, m$, an extension C_i that splits $p_i(X)$, because then $C = C_1 \otimes_{A_*} \dots \otimes_{A_*} C_m$ will split them all, so we can assume that $m = 1$ and $p_1(X) = p(X)$; moreover, by induction on the degree of $p(X)$, it suffices to find an extension C' such that $p(X)$ factors in $C'[X]$ as a product of the form $p(X) = (X - \alpha) \cdot q(X)$, where $q(X)$ is a monic polynomial of degree $\deg(p) - 1$. Clearly we can take $C' = A_*[T]/(p(T))$.

Given a C as in claim 3.4.25, we remark that the morphism $A \rightarrow C^a$ is of universal F -2-descent. Considering again the topology of universal F -2-descent, it follows that $A \rightarrow B$ is of universal F -2-descent if and only if the same holds for the induced morphism $C^a \rightarrow C^a \otimes_A B$. Therefore, in proving assertion (iii) we can replace ϕ by $\mathbf{1}_C \otimes_A \phi$ and assume from start that the polynomials $p_i(X)$ factor in $A_*[X]$ as product of linear factors. Now, let $\deg(p_i) = d_i$ and $p_i(X) = \prod_j^{d_i} (X - \alpha_{ij})$ (for $i = 1, \dots, m$). We get a surjective homomorphism of A_* -algebras $D = A_*[X_1, \dots, X_m]/(p_1(X_1), \dots, p_m(X_m)) \rightarrow R$ by the rule $X_i \mapsto f_i$ ($i = 1, \dots, m$). Moreover, any sequence $\underline{\alpha} = (\alpha_{1,j_1}, \alpha_{2,j_2}, \dots, \alpha_{m,j_m})$ yields a homomorphism $\psi_{\underline{\alpha}} : D \rightarrow A_*$, determined by the assignment $X_i \mapsto \alpha_{i,j_i}$. A simple combinatorial argument shows that $\prod_{\underline{\alpha}} \text{Ker}(\psi_{\underline{\alpha}}) = 0$, where $\underline{\alpha}$ runs over all the sequences as above. Hence the product map $\prod_{\underline{\alpha}} \psi_{\underline{\alpha}} : D \rightarrow \prod_{\underline{\alpha}} A_*$ has nilpotent kernel. We notice that the A_* -algebra $(\prod_{\underline{\alpha}} A_*) \otimes_D R$ is a quotient of $\prod_{\underline{\alpha}} A_*$, hence it

can be written as a product of rings of the form $A_*/I_{\underline{\alpha}}$, for various ideals $I_{\underline{\alpha}}$. By (i), the kernel of the induced homomorphism $R \rightarrow \prod_{\underline{\alpha}} A_*/I_{\underline{\alpha}}$ is nilpotent, hence the same holds for the kernel of the composition $A \rightarrow \prod_{\underline{\alpha}} A/I_{\underline{\alpha}}^a$, which is therefore of the kind considered in claim 3.4.24. Hence $A \rightarrow \prod_{\underline{\alpha}} A/I_{\underline{\alpha}}^a$ is of universal F -2-descent. Since such morphisms form a topology, it follows that also $A \rightarrow B$ is of universal F -2-descent, which concludes the proof of (iii).

Finally, let M be as in (ii) and pick again C as in the proof of claim 3.4.25. By remark 3.3.17(iv), M is almost projective over A if and only if $C^a \otimes_A M$ is almost projective over C^a ; hence we can replace ϕ by $\mathbf{1}_{C^a} \otimes_A \phi$, and by arguing as in the proof of (iii), we can assume from start that $B = \prod_{j=1}^n (A/I_j)$ for ideals $I_j \subset A$, $j = 1, \dots, n$ such that $I = \bigcap_{j=1}^n I_j$ is nilpotent. By an easy induction, we can furthermore reduce to the case $n = 2$. We factor ϕ as $A \rightarrow A/I \rightarrow B$; by proposition 3.4.11 it follows that $(A/I) \otimes_A M$ is almost projective over A/I , and then lemma 3.3.16(i) says that M itself is almost projective. \square

Remark 3.4.26. It is natural to ask whether theorem 3.4.22 holds if we replace everywhere “strictly finite” by “finite with nilpotent kernel” (or even by “almost finite with nilpotent kernel”). We do not know the answer to this question.

We conclude with a digression to explain the relationship between our results and related facts that can be extracted from the literature. So, we now place ourselves in the “classical limit” $V = \mathfrak{m}$ (cp. example 2.1.1(ii)). In this case, weakly étale morphisms had already been considered in some earlier work, and they were called “absolutely flat” morphisms. A ring homomorphism $A \rightarrow B$ is étale in the usual sense of [10] if and only if it is absolutely flat and of finite presentation. Let us denote by $\mathbf{u}\text{-}\acute{\text{E}}\text{t}^o$ the fibred category over $V\text{-}\mathbf{Alg}^o$, whose fibre over a V -algebra A is the opposite of the category of étale A -algebras in the usual sense. We claim that, if a morphism $A \rightarrow B$ of V -algebras is of universal effective descent for the fibred category $\mathbf{w}\text{-}\acute{\text{E}}\text{t}^o$ (resp. $\acute{\text{E}}\text{t}^o$), then it is a morphism of universal effective descent for $\mathbf{u}\text{-}\acute{\text{E}}\text{t}^o$. Indeed, let C be an étale A -algebra (in the sense of definition 3.1.1) and such that $C \otimes_A B$ is étale over B in the usual sense. We have to show that C is étale in the usual sense, *i.e.* that it is of finite presentation over A . This amounts to showing that, for every filtered inductive system $(A_\lambda)_{\lambda \in \Lambda}$ of A -algebras, we have $\text{colim}_{\lambda \in \Lambda} \text{Hom}_{A\text{-}\mathbf{Alg}}(C, A_\lambda) \simeq \text{Hom}_{A\text{-}\mathbf{Alg}}(C, \text{colim}_{\lambda \in \Lambda} A_\lambda)$. Since, by assumption, this is known after extending scalars to B and to $B \otimes_A B$, it suffices to show that, for any A -algebra D , the natural sequence

$$\text{Hom}_{A\text{-}\mathbf{Alg}}(C, D) \longrightarrow \text{Hom}_{B\text{-}\mathbf{Alg}}(C_B, D_B) \rightrightarrows \text{Hom}_{B \otimes_A B\text{-}\mathbf{Alg}}(C_{B \otimes_A B}, D_{B \otimes_A B})$$

is exact. For this, note that $\text{Hom}_{A\text{-}\mathbf{Alg}}(C, D) = \text{Hom}_{D\text{-}\mathbf{Alg}}(C_D, D)$ (and similarly for the other terms) and by hypothesis $(D \rightarrow D \otimes_A B)^o$ is a morphism of 1-descent for the fibred category $\mathbf{w}\text{-}\acute{\text{E}}\text{t}^o$ (resp. $\acute{\text{E}}\text{t}^o$).

As a consequence of these observations and of theorem 3.4.22, we see that any finite ring homomorphism $\phi : A \rightarrow B$ with nilpotent kernel is of universal effective descent for the fibred category of étale algebras. This fact was known as follows. By [10] (Exp.IX, 4.7), $\text{Spec}(\phi)$ is of universal effective descent for the fibred category of separated étale morphisms of finite type. One has to show that if X is such a scheme over A , such that $X \otimes_A B$ is affine, then X is affine. This follows by reduction to the noetherian case and [5] (Chap.II, 6.7.1).

3.5. Behaviour of étale morphisms under Frobenius. We consider the following category \mathcal{B} of base rings. The objects of \mathcal{B} are the pairs (V, \mathfrak{m}) , where V is a ring and \mathfrak{m} is an ideal of V with $\mathfrak{m} = \mathfrak{m}^2$ and $\widehat{\mathfrak{m}}$ is flat. The morphisms $(V, \mathfrak{m}_V) \rightarrow (W, \mathfrak{m}_W)$ between two objects of \mathcal{B} are the ring homomorphisms $f : V \rightarrow W$ such that $\mathfrak{m}_W = f(\mathfrak{m}_V) \cdot W$.

We have a fibred and cofibred category $\mathcal{B}\text{-}\mathbf{Mod} \rightarrow \mathcal{B}$ (see [10] (Exp.VI §5,6,10) for generalities on fibred categories). An object of $\mathcal{B}\text{-}\mathbf{Mod}$ (which we may call a “ \mathcal{B} -module”) consists of a pair $((V, \mathfrak{m}), M)$, where (V, \mathfrak{m}) is an object of \mathcal{B} and M is a V -module. Given two objects

$X = ((V, \mathfrak{m}_V), M)$ and $Y = ((W, \mathfrak{m}_W), N)$, the morphisms $X \rightarrow Y$ are the pairs (f, g) , where $f : (V, \mathfrak{m}_V) \rightarrow (W, \mathfrak{m}_W)$ is a morphism in \mathcal{B} and $g : M \rightarrow N$ is an f -linear map.

Similarly one has a fibred and cofibred category $\mathcal{B}\text{-Alg} \rightarrow \mathcal{B}$ of \mathcal{B} -algebras. We will also need to consider the fibred and cofibred category $\mathcal{B}\text{-Mon} \rightarrow \mathcal{B}$ of non-unitary commutative \mathcal{B} -monoids: an object of $\mathcal{B}\text{-Mon}$ is a pair $((V, \mathfrak{m}), A)$ where A is a V -module endowed with a morphism $A \otimes_V A \rightarrow A$ subject to associativity and commutativity conditions, as discussed in section 2.2. The fibre over an object (V, \mathfrak{m}) of \mathcal{B} , is the category of V -monoids denoted $(V, \mathfrak{m})\text{-Mon}$ or simply $V\text{-Mon}$.

The almost isomorphisms in the fibres of $\mathcal{B}\text{-Mod} \rightarrow \mathcal{B}$ give a multiplicative system Σ in $\mathcal{B}\text{-Mod}$, admitting a calculus of both left and right fractions. The “locally small” conditions are satisfied (see [22] p.381), so that one can form the localised category $\mathcal{B}^a\text{-Mod} = \Sigma^{-1}(\mathcal{B}\text{-Mod})$. The fibres of the localised category over the objects of \mathcal{B} are the previously considered categories of almost modules. Similar considerations hold for $\mathcal{B}\text{-Alg}$ and $\mathcal{B}\text{-Mon}$, and we get the fibred and cofibred categories $\mathcal{B}^a\text{-Mod} \rightarrow \mathcal{B}$, $\mathcal{B}^a\text{-Alg} \rightarrow \mathcal{B}$ and $\mathcal{B}^a\text{-Mon} \rightarrow \mathcal{B}$. In particular, for every object (V, \mathfrak{m}) of \mathcal{B} , we have an obvious notion of almost V -monoid and the category consisting of these is denoted $V^a\text{-Mon}$. The localisation functors

$$\mathcal{B}\text{-Mod} \rightarrow \mathcal{B}^a\text{-Mod} : M \mapsto M^a \quad \mathcal{B}\text{-Alg} \rightarrow \mathcal{B}^a\text{-Alg} : B \mapsto B^a$$

have left and right adjoints. These adjoints can be chosen as functors of categories over \mathcal{B} such that the adjunction units and counits are morphisms over identity arrows in \mathcal{B} . On the fibres these induce the previously considered left and right adjoints $M \mapsto M_!$, $M \mapsto M_*$, $B \mapsto B_!$, $B \mapsto B_*$. We will use the same notation for the corresponding functors on the larger categories. Then it is easy to check that the functor $M \mapsto M_!$ is cartesian and cocartesian (*i.e.* it sends cartesian arrows to cartesian arrows and cocartesian arrows to cocartesian arrows), $M \mapsto M_*$ and $B \mapsto B_*$ are cartesian, and $B \mapsto B_!$ is cocartesian.

Let \mathcal{B}/\mathbb{F}_p be the full subcategory of \mathcal{B} consisting of all objects (V, \mathfrak{m}) where V is an \mathbb{F}_p -algebra. Define similarly $\mathcal{B}\text{-Alg}/\mathbb{F}_p$, $\mathcal{B}\text{-Mon}/\mathbb{F}_p$ and $\mathcal{B}^a\text{-Alg}/\mathbb{F}_p$, $\mathcal{B}^a\text{-Mon}/\mathbb{F}_p$, so that we have again fibred and cofibred categories $\mathcal{B}^a\text{-Alg}/\mathbb{F}_p \rightarrow \mathcal{B}/\mathbb{F}_p$ and $\mathcal{B}^a\text{-Mon}/\mathbb{F}_p \rightarrow \mathcal{B}/\mathbb{F}_p$ (resp. the same for non-unitary monoids). We remark that the categories $\mathcal{B}^a\text{-Alg}/\mathbb{F}_p$ and $\mathcal{B}^a\text{-Mon}/\mathbb{F}_p$ have small limits and colimits, and these are preserved by the projection to \mathcal{B}/\mathbb{F}_p . Especially, if $A \rightarrow B$ and $A \rightarrow C$ are two morphisms in $\mathcal{B}^a\text{-Alg}/\mathbb{F}_p$ or $\mathcal{B}^a\text{-Mon}/\mathbb{F}_p$, we can define $B \otimes_A C$ as such a colimit.

If A is a (unitary or non-unitary) \mathcal{B} -monoid over \mathbb{F}_p , we denote by $\phi_A : A \rightarrow A$ the Frobenius endomorphism $x \mapsto x^p$. If (V, \mathfrak{m}) is an object of \mathcal{B}/\mathbb{F}_p , it follows from proposition 2.1.5(ii) that $\phi_V : (V, \mathfrak{m}) \rightarrow (V, \mathfrak{m})$ is a morphism in \mathcal{B} . If B is an object of $\mathcal{B}\text{-Alg}/\mathbb{F}_p$ (resp. $\mathcal{B}\text{-Mon}/\mathbb{F}_p$) over V , then the Frobenius map induces a morphism $\phi_B : B \rightarrow B$ in $\mathcal{B}\text{-Alg}/\mathbb{F}_p$ (resp. $\mathcal{B}\text{-Mon}/\mathbb{F}_p$) over ϕ_V . In this way we get a natural transformation from the identity functor of $\mathcal{B}\text{-Alg}/\mathbb{F}_p$ (resp. $\mathcal{B}\text{-Mon}/\mathbb{F}_p$) to itself that induces a natural transformation on the identity functor of $\mathcal{B}^a\text{-Alg}/\mathbb{F}_p$ (resp. $\mathcal{B}^a\text{-Mon}/\mathbb{F}_p$).

Using the pull-back functors, any object B of $\mathcal{B}\text{-Alg}$ over V defines new objects $B_{(m)}$ of $\mathcal{B}\text{-Alg}$ ($m \in \mathbb{N}$) over V , where $B_{(m)} = (\phi_V^m)^*(B)$, which is just B considered as a V -algebra via the homomorphism $V \xrightarrow{\phi^m} V \rightarrow B$. These operations also induce functors $B \mapsto B_{(m)}$ on almost \mathcal{B} -algebras.

Definition 3.5.1. i) Let (V, \mathfrak{m}) be an object of \mathcal{B}/\mathbb{F}_p ; we say that a morphism $f : A \rightarrow B$ of almost V -algebras (resp. almost V -monoids) is *invertible up to ϕ^m* if there exists a morphism $f' : B \rightarrow A$ in $\mathcal{B}^a\text{-Alg}$ (resp. $\mathcal{B}^a\text{-Mon}$) over ϕ_V^m , such that $f' \circ f = \phi_A^m$ and $f \circ f' = \phi_B^m$.

ii) We say that an almost V -monoid I (*e.g.* an ideal in a V^a -algebra) is *Frobenius nilpotent* if ϕ_I is nilpotent.

Notice that a morphism f of $V^a\text{-Alg}$ (or $V^a\text{-Mon}$) is invertible up to ϕ^m if and only if $f_* : A_* \rightarrow B_*$ is so as a morphism of \mathbb{F}_p -algebras.

Lemma 3.5.2. *Let (V, \mathfrak{m}) be an object of \mathcal{B}/\mathbb{F}_p and let $f : A \rightarrow B$, $g : B \rightarrow C$ be morphisms of almost V -algebras or almost V -monoids.*

- i) *If f is invertible up to ϕ^n and g is invertible up to ϕ^m , then $g \circ f$ is invertible up to ϕ^{m+n} .*
- ii) *If f is invertible up to ϕ^n and $g \circ f$ is invertible up to ϕ^m , then g is invertible up to ϕ^{m+n} .*
- iii) *If g is invertible up to ϕ^n and $g \circ f$ is invertible up to ϕ^m , then f is invertible up to ϕ^{m+n} .*
- iv) *The Frobenius morphisms induce ϕ_V -linear morphisms (i.e. morphisms in $\mathcal{B}^a\text{-Mod}$ over ϕ_V) $\phi' : \text{Ker}(f) \rightarrow \text{Ker}(f)$ and $\phi'' : \text{Coker}(f) \rightarrow \text{Coker}(f)$, and f is invertible up to some power of ϕ if and only if both ϕ' and ϕ'' are nilpotent.*
- v) *Consider a map of short exact sequences of almost V -monoids :*

$$\begin{array}{ccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' \longrightarrow 0 \\ & & \downarrow f' & & \downarrow f & & \downarrow f'' \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' \longrightarrow 0 \end{array}$$

and suppose that two of the morphisms f' , f , f'' are invertible up to a power of ϕ . Then also the third morphism has this property.

Proof. (i): if f' is an inverse of f up to ϕ^n and g' is an inverse of g up to ϕ^m , then $f' \circ g'$ is an inverse of $g \circ f$ up to ϕ^{m+n} . (ii): given an inverse f' of f up to ϕ^n and an inverse h' of $h = g \circ f$ up to ϕ^m , let $g' = \phi_B^n \circ f \circ h'$. We compute :

$$\begin{aligned} g \circ g' &= g \circ \phi_B^n \circ f \circ h' = \phi_C^n \circ g \circ f \circ h = \phi_C^n \circ \phi_C^m \\ g' \circ g &= \phi_B^n \circ f \circ h' \circ g = f \circ h' \circ g \circ \phi_B^n = f \circ h' \circ g \circ f \circ f' \\ &= f \circ \phi_A^m \circ f' = \phi_B^m \circ f \circ f' = \phi_B^m \circ \phi_B^n. \end{aligned}$$

(iii) is similar and (iv) is an easy diagram chasing left to the reader. (v) follows from (iv) and the snake lemma. \square

Lemma 3.5.3. *Let (V, \mathfrak{m}) be an object of \mathcal{B}/\mathbb{F}_p .*

- (i) *If $f : A \rightarrow B$ is a morphism of almost V -algebras, invertible up to ϕ^n , then so is $A' \rightarrow A' \otimes_A B$ for every morphism $A \rightarrow A'$ of almost algebras.*
- (ii) *If $f : (V, \mathfrak{m}_V) \rightarrow (W, \mathfrak{m}_W)$ is a morphism in \mathcal{B}/\mathbb{F}_p , the functors $f_* : (V, \mathfrak{m}_V)^a\text{-Alg} \rightarrow (W, \mathfrak{m}_W)^a\text{-Alg}$ and $f^* : (W, \mathfrak{m}_W)^a\text{-Alg} \rightarrow (V, \mathfrak{m}_V)^a\text{-Alg}$ preserve the class of morphisms invertible up to ϕ^n .*

Proof. (i): given $f' : B \rightarrow A_{(m)}$, construct a morphism $A' \otimes_A B \rightarrow A'_{(m)}$ using the morphism $A' \rightarrow A'_{(m)}$ coming from $\phi_{A'}^m$ and f' . (ii): the assertion for f^* is clear, and the assertion for f_* follows from (i). \square

Remark 3.5.4. Statements like those of lemma 3.5.3 hold for the classes of flat, (weakly) unramified, (weakly) étale morphisms.

Theorem 3.5.5. *Let (V, \mathfrak{m}) be an object of \mathcal{B}/\mathbb{F}_p and $f : A \rightarrow B$ a weakly étale morphism of almost V -algebras.*

- (i) *If f is invertible up to ϕ^n ($n \geq 0$), then it is an isomorphism.*
- (ii) *For every integer $m \geq 0$ the natural square diagram*

$$(3.5.6) \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ \phi_A^m \downarrow & & \downarrow \phi_B^m \\ A_{(m)} & \xrightarrow{f_{(m)}} & B_{(m)} \end{array}$$

is cocartesian.

Proof. (i): we first show that f is faithfully flat. Since f is flat, it remains to show that if M is an A -module such that $M \otimes_A B = 0$, then $M = 0$. It suffice to do this for $M = A/I$, for an arbitrary ideal I of A . After base change by $A \rightarrow A/I$, we reduce to show that $B = 0$ implies $A = 0$. However, $A_* \rightarrow B_*$ is invertible up to ϕ^n , so $\phi_{A_*}^n = 0$ which means $A_* = 0$. In particular, f is a monomorphism, hence the proof is complete in case that f is an epimorphism.

In general, consider the composition $B \xrightarrow{1_B \otimes f} B \otimes_A B \xrightarrow{\mu_{B/A}} B$. From lemma 3.5.3(i) it follows that $1_B \otimes f$ is invertible up to ϕ^n ; then lemma 3.5.2(ii) says that $\mu_{B/A}$ is invertible up to ϕ^n . The latter is also weakly étale; by the foregoing we derive that it is an isomorphism. Consequently $1_B \otimes f$ is an isomorphism, and finally, by faithful flatness, f itself is an isomorphism.

(ii): the morphisms ϕ_A^m and ϕ_B^m are invertible up to ϕ^m . By lemma 3.5.3(i) it follows that $1_B \otimes \phi_A^m : B \rightarrow B \otimes_A A_{(m)}$ is invertible up to ϕ^m ; hence, by lemma 3.5.2(ii), the morphism $h : B \otimes_A A_{(m)} \rightarrow B_{(m)}$ induced by ϕ_B^m and $f_{(m)}$ is invertible up to ϕ^{2m} (in fact one verifies that it is invertible up to ϕ^m). But h is a morphism of weakly étale $A_{(m)}$ -algebras, so it is weakly étale, so it is an isomorphism by (i). \square

Remark 3.5.7. Theorem 3.5.5(ii) extends a statement of Faltings ([6] p.10) for his notion of almost étale extensions.

We recall (cp. [9] (Chap.0, 3.5)) that a morphism $f : X \rightarrow Y$ of objects in a site is called *bicovering* if the induced map of associated sheaves of sets is an isomorphism; if f is squarable (“quarrable” in French), this is equivalent to the condition that both f and the diagonal morphism $X \rightarrow X \times_Y X$ are covering morphisms.

Let $F \rightarrow E$ be a fibered category and $f : P \rightarrow Q$ a squarable morphism of E . Consider the following condition:

(3.5.8) for every base change $P \times_Q Q' \rightarrow Q'$ of f , the inverse image functor $F_{Q'} \rightarrow F_{P \times_Q Q'}$ is an equivalence of categories.

Inspecting the arguments in [9] (Chap.II, §1.1) one can show:

Lemma 3.5.9. *With the above notation, let τ be the topology of universal effective descent relative to $F \rightarrow E$. Then we have :*

- i) if (3.5.8) holds, then f is a covering morphism for the topology τ ;
- ii) f is bicovering for τ if and only if (3.5.8) holds both for f and for the diagonal morphism $P \rightarrow P \times_Q P$.

Remark 3.5.10. In [9] (Chap.II, 1.1.3(iv)) it is stated that “la réciproque est vraie si $i = 2$ ”, meaning that (3.5.8) is equivalent to the condition that f is bicovering for τ . (Actually the cited statement is given in terms of presheaves, but one can show that (3.5.8) is equivalent to the corresponding condition for the fibered category $F^+ \rightarrow \widehat{E}_U$ considered in *op.cit.*) However, this fails in general : as a counterexample we can give the following. Let E be the category of schemes of finite type over a field k ; set $P = \mathbb{A}_k^1$, $Q = \text{Spec}(k)$. Finally let $F \rightarrow E$ be the discretely fibered category defined by the presheaf $X \mapsto H^0(X, \mathbb{Z})$. Then it is easy to show that f satisfies (3.5.8) but the diagonal map does not, so f is not bicovering. The mistake in the proof is in [9] (Chap.II, 1.1.3.5), where one knows that $F^+(d)$ is an equivalence of categories (notation of *loc.cit.*) but one needs it also after base changes of d .

Lemma 3.5.11. (i) *If $f : A \rightarrow B$ is a morphism of V^a -algebras which is invertible up to ϕ^m , then the induced functors $\dot{\text{Ét}}(A) \rightarrow \dot{\text{Ét}}(B)$ and $\mathbf{w}.\dot{\text{Ét}}(A) \rightarrow \mathbf{w}.\dot{\text{Ét}}(B)$ are equivalences of categories.*

ii) *If $A \rightarrow B$ is weakly étale and $C \rightarrow D$ is a morphism of A -algebras invertible up to ϕ^m , then the induced map $\text{Hom}_{A\text{-Alg}}(B, C) \rightarrow \text{Hom}_{A\text{-Alg}}(B, D)$ is bijective.*

Proof. We first consider (i) for the special case where $f = \phi_A^m : A \rightarrow A_{(m)}$. The functor $(\phi_V^m)^* : V^a\text{-Alg} \rightarrow V^a\text{-Alg}$ induces a functor $(-)_{(m)} : A\text{-Alg} \rightarrow A_{(m)}\text{-Alg}$, and by restriction

(see remark 3.5.3) we obtain a functor $(-)_m : \dot{\mathbf{E}}\mathbf{t}(A) \rightarrow \dot{\mathbf{E}}\mathbf{t}(A_{(m)})$; by theorem 3.5.5(ii), the latter is isomorphic to the functor $(\phi^m)_* : \dot{\mathbf{E}}\mathbf{t}(A) \rightarrow \dot{\mathbf{E}}\mathbf{t}(A_{(m)})$ of the lemma. Furthermore, from remark 2.1.3(ii) and (2.2.2) we derive a natural ring isomorphism $\omega : A_{(m)*} \simeq A_*$, hence an essentially commutative diagram

$$\begin{array}{ccccc} \dot{\mathbf{E}}\mathbf{t}(A) & \longrightarrow & A\text{-}\mathbf{Alg} & \xrightarrow{\alpha} & (A_*, \mathfrak{m} \cdot A_*)^a\text{-}\mathbf{Alg} \\ (\phi^m)_* \downarrow & & (-)_m \downarrow & & \omega^* \downarrow \\ \dot{\mathbf{E}}\mathbf{t}(A_{(m)}) & \longrightarrow & A_{(m)}\text{-}\mathbf{Alg} & \xrightarrow{\beta} & (A_{(m)*}, \mathfrak{m} \cdot A_{(m)*})^a\text{-}\mathbf{Alg} \end{array}$$

where α and β are the equivalences of remark 2.2.3. Clearly α and β restrict to equivalences on the corresponding categories of étale algebras, hence the lemma follows in this case.

For the general case of (i), let $f' : B \rightarrow A_{(m)}$ be a morphism as in definition 3.5.1. Diagram (3.5.6) induces an essentially commutative diagram of the corresponding categories of algebras, so by the previous case, the functor $(f')_* : \dot{\mathbf{E}}\mathbf{t}(B) \rightarrow \dot{\mathbf{E}}\mathbf{t}(A_{(m)})$ has both a left essential inverse and a right essential inverse; these essential inverses must be isomorphic, so f_* has an essential inverse as desired. Finally, we remark that the map in (ii) is the same as the map $\mathrm{Hom}_{C\text{-}\mathbf{Alg}}(B \otimes_A C, C) \rightarrow \mathrm{Hom}_{D\text{-}\mathbf{Alg}}(B \otimes_A D, D)$, and the latter is a bijection in view of (i). \square

Remark 3.5.12. Notice that lemma 3.5.11(ii) generalises the lifting theorem 3.3.12(i) (in case V is an \mathbb{F}_p -algebra). Similarly, it follows from lemmata 3.5.11(i) and 3.5.2(iv) that, in case V is an \mathbb{F}_p -algebra, one can replace “nilpotent” in theorem 3.3.12 parts (ii) and (iii) by “Frobenius nilpotent”.

In the following τ will denote indifferently the topology of universal effective descent defined by either of the fibered categories $\mathbf{w}\dot{\mathbf{E}}\mathbf{t}^o \rightarrow V^a\text{-}\mathbf{Alg}^o$ or $\dot{\mathbf{E}}\mathbf{t}^o \rightarrow V^a\text{-}\mathbf{Alg}^o$.

Proposition 3.5.13. *If $f : A \rightarrow B$ is a morphism of almost V -algebras which is invertible up to ϕ^m , then f^o is biconverging for the topology τ .*

Proof. In light of lemmata 3.5.9(ii) and 3.5.11(i), it suffices to show that $\mu_{B/A}$ is invertible up to a power of ϕ . For this, factor the identity morphism of B as $B \xrightarrow{1_B \otimes f} B \otimes_A B \xrightarrow{\mu_{B/A}} B$ and argue as in the proof of theorem 3.5.5. \square

Proposition 3.5.14. *Let $A \rightarrow B$ be a morphism of almost V -algebras and $I \subset A$ an ideal. Set $\overline{A} = A/I$ and $\overline{B} = B/I \cdot B$. Suppose that either*

- a) *$I \rightarrow I \cdot B$ is an epimorphism with nilpotent kernel, or*
- b) *V is an \mathbb{F}_p -algebra and $I \rightarrow I \cdot B$ is invertible up to a power of ϕ .*

Then we have :

- i) *conditions (a) and (b) are stable under any base change $A \rightarrow C$.*
- ii) *$(A \rightarrow B)^o$ is covering (resp. biconverging) for τ if and only if $(\overline{A} \rightarrow \overline{B})^o$ is.*

Proof. Suppose first that $I \rightarrow I \cdot B$ is an isomorphism; in this case we claim that $I \cdot C \rightarrow I \cdot (C \otimes_A B)$ is an epimorphism and $\mathrm{Ker}(I \cdot C \rightarrow I \cdot (C \otimes_A B))^2 = 0$ for any A -algebra C . Indeed, since by assumption $I \simeq I \cdot B$, $C \otimes_A B$ acts on $C \otimes_A I$, hence $\mathrm{Ker}(C \rightarrow C \otimes_A B)$ annihilates $C \otimes_A I$, hence annihilates its image $I \cdot C$, whence the claim. If, moreover, V is an \mathbb{F}_p -algebra, lemma 3.5.2(iv) implies that $I \cdot C \rightarrow I \cdot (C \otimes_A B)$ is invertible up to a power of ϕ .

In the general case, consider the intermediate almost V -algebra $A_1 = \overline{A} \times_{\overline{B}} B$ equipped with the ideal $I_1 = 0 \times_{\overline{B}} (I \cdot B)$. In case (a), $I_1 = I \cdot A_1$ and $A \rightarrow A_1$ is an epimorphism with nilpotent kernel, hence it remains such after any base change $A \rightarrow C$. To prove (i) in case (a), it suffices then to consider the morphism $A_1 \rightarrow B$, hence we can assume from start that $I \rightarrow I \cdot B$ is an isomorphism, which is the case already dealt with. To prove (i) in case (b), it suffices to consider the cases of $(A, I) \rightarrow (A_1, I_1)$ and $(A_1, I_1) \rightarrow (B, I \cdot B)$. The second case is treated

above. In the first case, we do not necessarily have $I_1 = I \cdot A_1$ and the assertion to be checked is that, for every A -algebra C , the morphism $I \cdot C \rightarrow I_1 \cdot (A_1 \otimes_A C)$ is invertible up to a power of ϕ . We apply lemma 3.5.2(v) to the commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & I \cdot B & \longrightarrow & A_1 & \longrightarrow & A/I \longrightarrow 0 \end{array}$$

to deduce that $A \rightarrow A_1$ is invertible up to some power of ϕ , hence so is $C \rightarrow A_1 \otimes_A C$, which implies the assertion.

As for (ii), we remark that the “only if” part is trivial; and we assume therefore that $(\overline{A} \rightarrow \overline{B})^\circ$ is τ -covering (resp. τ -bicovering). Consider first the assertion for “covering”. We need to show that $(A \rightarrow B)^\circ$ is of universal effective descent for F , where F is either one of our two fibered categories. In light of (i), this is reduced to the assertion that $(A \rightarrow B)^\circ$ is of effective descent for F . We notice that $(A \rightarrow A_1)^\circ$ is bicovering for τ (in case (a) by theorem 3.3.12 and lemma 3.5.9(ii), in case (b) by proposition 3.5.13). As $(\overline{A} \rightarrow A_1/I_1)^\circ$ is an isomorphism, the assertion is reduced to the case where $I \rightarrow I \cdot B$ is an isomorphism. In this case, by lemma 3.4.16, there is a natural equivalence: $\mathbf{Desc}(F, B/A) \xrightarrow{\sim} \mathbf{Desc}(F, \overline{B}/\overline{A}) \times_{F_{\overline{B}}} F_B$. Then the assertion follows easily from corollary 3.4.13. Finally suppose that $(\overline{A} \rightarrow \overline{B})^\circ$ is bicovering. The foregoing already says that $(A \rightarrow B)^\circ$ is covering, so it remains to show that $(B \otimes_A B \rightarrow B)^\circ$ is also covering. The above argument again reduces to the case where $I \rightarrow I \cdot B$ is an isomorphism. Then, as in the proof of lemma 3.4.16, the induced morphism $I \cdot (B \otimes_A B) \rightarrow I \cdot B$ is an isomorphism as well. Thus the assertion for “bicovering” is reduced to the assertion for “covering”. \square

4. APPENDIX

4.1. In this appendix we have gathered a few miscellaneous results that were found in the course of our investigation, and which may be useful for other applications.

We need some preliminaries on simplicial objects : first of all, a simplicial almost algebra is just an object in the category $s.(V^a\text{-Alg})$. Then for a given simplicial almost algebra A we have the category $A\text{-Mod}$ of A -modules : it consists of all simplicial almost V -modules M such that $M[n]$ is an $A[n]$ -module and such that the face and degeneracy morphisms $d_i : M[n] \rightarrow M[n-1]$ and $s_i : M[n] \rightarrow M[n+1]$ ($i = 0, 1, \dots, n$) are $A[n]$ -linear. We will need also the derived category of A -modules; it is defined as follows.

A bit more generally, let \mathcal{C} be any abelian category. For an object X of $s.\mathcal{C}$ let $N(X)$ be the normalized chain complex (defined as in [14] (I.1.3)). By the theorem of Dold-Kan ([22] th.8.4.1) $X \mapsto N(X)$ induces an equivalence $N : s.\mathcal{C} \rightarrow \mathbf{C}_\bullet(\mathcal{C})$. Now we say that a morphism $X \rightarrow Y$ in $s.\mathcal{C}$ is a *quasi-isomorphism* if the induced morphism $N(X) \rightarrow N(Y)$ is a quasi-isomorphism of chain complexes.

In the following we fix a simplicial almost algebra A .

Definition 4.1.1. We say that A is *exact* if the almost algebras $A[n]$ are exact for all $n \in \mathbb{N}$. A morphism $\phi : M \rightarrow N$ of A -modules (or A -algebras) is a *quasi-isomorphism* if the morphism ϕ of underlying simplicial almost V -modules is a quasi-isomorphism. We define the category $\mathbf{D}_\bullet(A)$ (resp. the category $\mathbf{D}_\bullet(A\text{-Alg})$) as the localization of the category $A\text{-Mod}$ (resp. $A\text{-Alg}$) with respect to the class of quasi-isomorphisms.

As usual, the morphisms in $\mathbf{D}_\bullet(A)$ can be computed via a calculus of fraction on the category $\mathbf{Hot}_\bullet(A)$ of simplicial complexes up to homotopy. Moreover, if A_1 and A_2 are two simplicial

almost algebras, then the “extension of scalars” functors define equivalences of categories

$$\begin{aligned} \mathbf{D}_\bullet(A_1 \times A_2) &\xrightarrow{\sim} \mathbf{D}_\bullet(A_1) \times \mathbf{D}_\bullet(A_2) \\ \mathbf{D}_\bullet(A_1 \times A_2\text{-Alg}) &\xrightarrow{\sim} \mathbf{D}_\bullet(A_1\text{-Alg}) \times \mathbf{D}_\bullet(A_2\text{-Alg}). \end{aligned}$$

Proposition 4.1.2. (i) *The functor on A -algebras given by $B \mapsto (s.V^a \times B)_\parallel$ preserves quasi-isomorphisms and therefore induces a functor $\mathbf{D}_\bullet(A\text{-Alg}) \rightarrow \mathbf{D}_\bullet((s.V^a \times A)_\parallel\text{-Alg})$.*

(ii) *The localisation functor $R \mapsto R^a$ followed by “extension of scalars” via $s.V^a \times A \rightarrow A$ induces a functor $\mathbf{D}_\bullet((s.V^a \times A)_\parallel\text{-Alg}) \rightarrow \mathbf{D}_\bullet(A\text{-Alg})$ and the composition of this and the above functor is naturally isomorphic to the identity functor on $\mathbf{D}_\bullet(A\text{-Alg})$.*

Proof. (i) : let $B \rightarrow C$ be a quasi-isomorphism of A -algebras. Clearly the induced morphism $s.V^a \times B \rightarrow s.V^a \times C$ is still a quasi-isomorphism of V -algebras. But by remark 2.2.13, $s.V^a \times B$ and $s.V^a \times C$ are exact simplicial almost V -algebras; moreover, it follows from corollary 2.2.10 that $(s.V^a \times B)_\parallel \rightarrow (s.V^a \times C)_\parallel$ is a quasi-isomorphism of V -modules. Then the claim follows easily from the exactness of the sequence (2.2.11). Now (ii) is clear. \square

Remark 4.1.3. In case m is flat, then all A -algebras are exact, and the same argument shows that the functor $B \mapsto B_\parallel$ induces a functor $\mathbf{D}_\bullet(A\text{-Alg}) \rightarrow \mathbf{D}_\bullet(A_\parallel\text{-Alg})$. In this case, composition with localisation is naturally isomorphic to the identity functor on $\mathbf{D}_\bullet(A\text{-Alg})$.

Proposition 4.1.4. *Let $f : R \rightarrow S$ be a map of V -algebras such that $f^a : R^a \rightarrow S^a$ is an isomorphism. Then $\mathbb{L}_{S/R}^a \simeq 0$ in $\mathbf{D}_\bullet(s.S^a)$.*

Proof. We show by induction on q that

$$\mathbf{VAN}(q; S/R) \quad H_q(\mathbb{L}_{S/R}^a) = 0.$$

For $q = 0$ the claim follows immediately from [14] (II.1.2.4.2). Therefore suppose that $q > 0$ and that $\mathbf{VAN}(j; D/C)$ is known for all almost isomorphisms of V -algebras $C \rightarrow D$ and all $j < q$. Let $\overline{R} = f(R)$. Then by transitivity ([14] (II.2.1.2)) we have a distinguished triangle in $\mathbf{D}_\bullet(s.S^a)$

$$(S \otimes_{\overline{R}} \mathbb{L}_{\overline{R}/R})^a \xrightarrow{u} \mathbb{L}_{S/R}^a \xrightarrow{v} \mathbb{L}_{S/\overline{R}}^a \longrightarrow \sigma(S \otimes_{\overline{R}} \mathbb{L}_{\overline{R}/R})^a.$$

We deduce that $\mathbf{VAN}(q; \overline{R}/R)$ and $\mathbf{VAN}(q; S/\overline{R})$ imply $\mathbf{VAN}(q; S/R)$, thus we can assume that f is either injective or surjective. Let $S_\bullet \rightarrow S$ be the simplicial V -algebra augmented over S defined by $S_\bullet = P_V(S)$. It is a simplicial resolution of S by free V -algebras, in particular the augmentation is a quasi-isomorphism of simplicial V -algebras. Set $R_\bullet = S_\bullet \times_S R$. This is a simplicial V -algebra augmented over R via a quasi-isomorphism. Moreover, the induced morphisms $R[n]^a \rightarrow S[n]^a$ are isomorphisms. By [14] (II.1.2.6.2) there is a quasi-isomorphism $\mathbb{L}_{S/R} \simeq \mathbb{L}_{S_\bullet/R_\bullet}^\Delta$. On the other hand we have a spectral sequence

$$E_{ij}^1 = H_j(\mathbb{L}_{S[i]/R[i]}) \Rightarrow H_{i+j}(\mathbb{L}_{S_\bullet/R_\bullet}^\Delta).$$

It follows easily that $\mathbf{VAN}(j; S[i]/R[i])$ for all $i \geq 0, j \leq q$ implies $\mathbf{VAN}(q; S/R)$. Therefore we are reduced to the case where S is a free V -algebra and f is either injective or surjective. We examine separately these two cases. If $f : R \rightarrow V[T]$ is surjective, then we can find a right inverse $s : V[T] \rightarrow R$ for f . By applying transitivity to the sequence $V[T] \rightarrow R \rightarrow V[T]$ we get a distinguished triangle

$$(V[T] \otimes_R \mathbb{L}_{R/V[T]})^a \xrightarrow{u} \mathbb{L}_{V[T]/V[T]}^a \xrightarrow{v} \mathbb{L}_{V[T]/R}^a \rightarrow \sigma(V[T] \otimes_R \mathbb{L}_{R/V[T]})^a.$$

Since $\mathbb{L}_{V[T]/V[T]}^a \simeq 0$ there follows an isomorphism : $H_q(\mathbb{L}_{V[T]/R}^a) \simeq H_{q-1}(V[T] \otimes_R \mathbb{L}_{R/V[T]})^a$. Furthermore, since f^a is an isomorphism, s^a is an isomorphism as well, hence by induction (and by a spectral sequence of the type [14] (I.3.3.3.2)) $H_{q-1}(V[T] \otimes_R \mathbb{L}_{R/V[T]})^a \simeq 0$. The claim follows in this case.

Finally suppose that $f : R \rightarrow V[T]$ is injective. Write $V[T] = \text{Sym}(F)$, for a free V -module F and set $\tilde{F} = \tilde{m} \otimes_V F$; since f^a is an isomorphism, $\text{Im}(\text{Sym}(\tilde{F}) \rightarrow \text{Sym}(F)) \subset R$. We apply transitivity to the sequence $\text{Sym}(\tilde{F}) \rightarrow R \rightarrow \text{Sym}(F)$. By arguing as above we are reduced to showing that $\mathbb{L}_{\text{Sym}(F)/\text{Sym}(\tilde{F})}^a \simeq 0$. We know that $H_0(\mathbb{L}_{\text{Sym}(F)/\text{Sym}(\tilde{F})}^a) \simeq 0$ and we will show that $H_q(\mathbb{L}_{\text{Sym}(F)/\text{Sym}(\tilde{F})}^a) \simeq 0$ for $q > 0$. To this purpose we apply transitivity to the sequence $V \rightarrow \text{Sym}(\tilde{F}) \rightarrow \text{Sym}(F)$. As F and \tilde{F} are flat V -modules, [14] (II.1.2.4.4) yields $H_q(\mathbb{L}_{\text{Sym}(F)/V}) \simeq H_q(\mathbb{L}_{\text{Sym}(\tilde{F})/V}) \simeq 0$ for $q > 0$ and $H_0(\mathbb{L}_{\text{Sym}(\tilde{F})/V})$ is a flat $\text{Sym}(\tilde{F})$ -module. In particular $H_j(\text{Sym}(F) \otimes_{\text{Sym}(\tilde{F})} \mathbb{L}_{\text{Sym}(\tilde{F})/V}) \simeq 0$ for all $j > 0$. Consequently $H_{j+1}(\mathbb{L}_{\text{Sym}(F)/\text{Sym}(\tilde{F})}) \simeq 0$ for all $j > 0$ and $H_1(\mathbb{L}_{\text{Sym}(F)/\text{Sym}(\tilde{F})}) \simeq \text{Ker}(\text{Sym}(F) \otimes_{\text{Sym}(\tilde{F})} \Omega_{\text{Sym}(\tilde{F})/V} \rightarrow \Omega_{\text{Sym}(F)/V})$. The latter module is easily seen to be almost zero. \square

Theorem 4.1.5. *Let $\phi : R \rightarrow S$ be a map of simplicial V -algebras inducing an isomorphism $R^a \xrightarrow{\sim} S^a$ in $\mathbf{D}_\bullet(R^a)$. Then $(\mathbb{L}_{S/R}^\Delta)^a \simeq 0$ in $\mathbf{D}_\bullet(S^a)$.*

Proof. Apply the base change theorem ([14] II.2.2.1) to the (flat) projections of $s.V \times R$ onto R and respectively $s.V$ to deduce that the natural map $\mathbb{L}_{s.V \times S/s.V \times R}^\Delta \rightarrow \mathbb{L}_{S/R}^\Delta \oplus \mathbb{L}_{s.V/s.V}^\Delta \rightarrow \mathbb{L}_{S/R}^\Delta$ is a quasi-isomorphism in $\mathbf{D}_\bullet(s.V \times S)$. By proposition 4.1.2 the induced morphism $(s.V \times R)_\#^a \rightarrow (s.V \times S)_\#^a$ is still a quasi-isomorphism. There are spectral sequences

$$\begin{aligned} E_{ij}^1 &= H_j(\mathbb{L}_{(V \times R[i])/(V \times R[i])_\#^a}) \Rightarrow H_{i+j}(\mathbb{L}_{(s.V \times R)/(s.V \times R)_\#^a}^\Delta) \\ F_{ij}^1 &= H_j(\mathbb{L}_{(V \times S[i])/(V \times S[i])_\#^a}) \Rightarrow H_{i+j}(\mathbb{L}_{(s.V \times S)/(s.V \times S)_\#^a}^\Delta). \end{aligned}$$

On the other hand, by proposition 4.1.4 we have $\mathbb{L}_{(V \times R[i])/(V \times R[i])_\#^a}^a \simeq 0 \simeq \mathbb{L}_{(V \times S[i])/(V \times S[i])_\#^a}^a$ for all $i \in \mathbb{N}$. Then the theorem follows directly from [14] (II.1.2.6.2(b)) and transitivity. \square

Proposition 4.1.6. *Let $A \rightarrow B$ be a morphism of exact almost V -algebras. Then the natural map $\tilde{m} \otimes_V \mathbb{L}_{B_\#/A_\#} \rightarrow \mathbb{L}_{B_\#/A_\#}$ is a quasi-isomorphism.*

Proof. By transitivity we may assume $A = V^a$. Let $P_\bullet = P_V(B_\#)$ be the standard resolution of $B_\#$ (see [14] II.1.2.1). Each $P[n]^a$ contains V as a direct summand, hence it is exact, so that we have an exact sequence of simplicial V -modules $0 \rightarrow s.\tilde{m} \rightarrow s.V \oplus (P_\bullet^a)_! \rightarrow (P_\bullet^a)_\# \rightarrow 0$. The augmentation $(P_\bullet^a)_! \rightarrow (B_\#^a)_! \simeq B_!$ is a quasi-isomorphism and we deduce that $(P_\bullet^a)_\# \rightarrow B_\#$ is a quasi-isomorphism; hence $(P_\bullet^a)_\# \rightarrow P_\bullet$ is a quasi-isomorphism as well. We have $P[n] \simeq \text{Sym}(F_n)$ for a free V -module F_n and the map $(P[n]^a)_\# \rightarrow P[n]$ is identified with $\text{Sym}(\tilde{m} \otimes_V F_n) \rightarrow \text{Sym}(F_n)$, whence $\Omega_{P[n]^a/V} \otimes_{P[n]^a} P[n] \rightarrow \Omega_{P[n]/V}$ is identified with $\tilde{m} \otimes_V \Omega_{P[n]/V} \rightarrow \Omega_{P[n]/V}$. By [14] (II.1.2.6.2) the map $\mathbb{L}_{(P_\bullet^a)_\#/V}^\Delta \rightarrow \mathbb{L}_{P_\bullet/V}^\Delta$ is a quasi-isomorphism. In view of [14] (II.1.2.4.4) we derive that $\Omega_{(P_\bullet^a)_\#/V} \rightarrow \Omega_{P_\bullet/V}$ is a quasi-isomorphism, i.e. $\tilde{m} \otimes_V \Omega_{P_\bullet/V} \rightarrow \Omega_{P_\bullet/V}$ is a quasi-isomorphism. Since \tilde{m} is flat and $\Omega_{P_\bullet/V} \rightarrow \Omega_{P_\bullet/V} \otimes_{P_\bullet} B_\# = \mathbb{L}_{B_\#/V}$ is a quasi-isomorphism, we get the desired conclusion. \square

In view of proposition 4.1.4 we have $\mathbb{L}_{(V^a \times A)_\#/V \times A_\#}^a \simeq 0$ in $\mathbf{D}_\bullet(V^a \times A)$. By this, transitivity and localisation ([14] II.2.3.1.1) we derive that $\mathbb{L}_{B/A}^a \rightarrow \mathbb{L}_{B_\#/A_\#}^a$ is a quasi-isomorphism for all A -algebras B . If A and B are exact (e.g. if \tilde{m} is flat), we conclude from proposition 4.1.6 that the natural map $\mathbb{L}_{B/A} \rightarrow \mathbb{L}_{B_\#/A_\#}$ is a quasi-isomorphism.

Finally we want to discuss left derived functors of (the almost version of) some notable non-additive functors that play a role in deformation theory. Let R be a simplicial V -algebra. Then we have an obvious functor $G : \mathbf{D}_\bullet(R) \rightarrow \mathbf{D}_\bullet(R^a)$ obtained by applying dimension-wise the localisation functor. Let Σ be the multiplicative set of morphisms of $\mathbf{D}_\bullet(R)$ that induce almost isomorphisms on the cohomology modules. An argument as in section 2.3 shows that G induces an equivalence of categories $\Sigma^{-1}\mathbf{D}_\bullet(R) \rightarrow \mathbf{D}_\bullet(R^a)$.

Now let R be a V -algebra and \mathcal{F}_p one of the functors \otimes^p , Λ^p , Sym^p , Γ^p defined in [14] (I.4.2.2.6).

Lemma 4.1.7. *Let $\phi : M \rightarrow N$ be an almost isomorphism of R -modules. Then $\mathcal{F}_p(\phi) : \mathcal{F}_p(M) \rightarrow \mathcal{F}_p(N)$ is an almost isomorphism.*

Proof. Let $\psi : \tilde{\mathfrak{m}} \otimes_V N \rightarrow M$ be the map corresponding to $(\phi^a)^{-1}$ under the bijection (2.2.2). By inspection, the compositions $\phi \circ \psi : \tilde{\mathfrak{m}} \otimes_V N \rightarrow N$ and $\psi \circ (\mathbf{1}_{\tilde{\mathfrak{m}}} \otimes \phi) : \tilde{\mathfrak{m}} \otimes_V M \rightarrow M$ are induced by scalar multiplication. Pick any $s \in \mathfrak{m}$ and lift it to an element $\tilde{s} \in \tilde{\mathfrak{m}}$; define $\psi_s : N \rightarrow M$ by $n \mapsto \psi(\tilde{s} \otimes n)$ for all $n \in N$. Then $\phi \circ \psi_s = s \cdot \mathbf{1}_N$ and $\psi_s \circ \phi = s \cdot \mathbf{1}_M$. This easily implies that s^p annihilates $\text{Ker} \mathcal{F}_p(\phi)$ and $\text{Coker} \mathcal{F}_p(\phi)$. In light of proposition 2.1.5(ii), the claim follows. \square

Let B be an almost V -algebra. We define a functor \mathcal{F}_p^a on $B\text{-Mod}$ by $M \mapsto (\mathcal{F}_p(M_!))^a$, where $M_!$ is viewed as a $B_{!!}$ -module or a B_* -module (to show that these choices define the same functor it suffices to observe that $B_* \otimes_{B_{!!}} N \simeq N$ for all B_* -modules N such that $N = \mathfrak{m} \cdot N$). For all $p > 0$ we have diagrams :

$$(4.1.8) \quad \begin{array}{ccc} R\text{-Mod} & \xrightarrow{\mathcal{F}_p} & R\text{-Mod} \\ \updownarrow & & \updownarrow \\ R^a\text{-Mod} & \xrightarrow{\mathcal{F}_p^a} & R^a\text{-Mod} \end{array}$$

where the downward arrows are localisation and the upward arrows are the functors $M \mapsto M_!$. Lemma 4.1.7 implies that the downward arrows in the diagram commute (up to a natural isomorphism) with the horizontal ones. It will follow from the following proposition 4.1.9 that the diagram commutes also going upward.

For any V -module N we have an exact sequence $\Gamma^2 N \rightarrow \otimes^2 N \rightarrow \Lambda^2 N \rightarrow 0$. As observed in the proof of proposition 2.1.5, the symmetric group S_2 acts trivially on $\otimes^2 \tilde{\mathfrak{m}}$ and $\Gamma^2 \tilde{\mathfrak{m}} \simeq \otimes^2 \tilde{\mathfrak{m}}$, so $\Lambda^2 \tilde{\mathfrak{m}} = 0$. Also we have natural isomorphisms $\Gamma^p \tilde{\mathfrak{m}} \simeq \tilde{\mathfrak{m}}$ for all $p > 0$.

Proposition 4.1.9. *Let R be a commutative ring and L a flat R -module with $\Lambda^2 L = 0$. Then for $p > 0$ and for all R -modules N we have natural isomorphisms*

$$\Gamma^p(L) \otimes_R \mathcal{F}_p(N) \xrightarrow{\sim} \mathcal{F}_p(L \otimes_R N).$$

Proof. Fix an element $x \in \mathcal{F}_p(N)$. For each R -algebra R' and each element $l \in R' \otimes_R L$ we get a map $\phi_l : R' \otimes_R N \rightarrow R' \otimes_R L \otimes_R N$ by $y \mapsto l \otimes y$, hence a map $\mathcal{F}_p(\phi_l) : R' \otimes_R \mathcal{F}_p(N) \simeq \mathcal{F}_p(R' \otimes_R N) \rightarrow \mathcal{F}_p(R' \otimes_R L \otimes_R N) \simeq R' \otimes_R \mathcal{F}_p(L \otimes_R N)$. For varying l we obtain a map of sets $\psi_{R',x} : R' \otimes_R L \rightarrow R' \otimes_R \mathcal{F}_p(L \otimes_R N) : l \mapsto \mathcal{F}_p(\phi_l)(1 \otimes x)$. According to the terminology of [21], the system of maps $\psi_{R',x}$ for R' ranging over all R -algebras forms a homogeneous polynomial law of degree p from L to $\mathcal{F}_p(L \otimes_R N)$, so it factors through the universal homogeneous degree p polynomial law $\gamma_p : L \rightarrow \Gamma^p(L)$. The resulting R -linear map $\bar{\psi}_x : \Gamma^p(L) \rightarrow \mathcal{F}_p(L \otimes_R N)$ depends R -linearly on x , hence we derive an R -linear map $\psi : \Gamma^p(L) \otimes_R \mathcal{F}_p(N) \rightarrow \mathcal{F}_p(L \otimes_R N)$. Next notice that by hypothesis S_2 acts trivially on $\otimes^2 L$ so S_p acts trivially on $\otimes^p L$ and we get an isomorphism $\beta : \Gamma^p(L) \xrightarrow{\sim} \otimes^p L$. We deduce a natural map $(\otimes^p L) \otimes_R \mathcal{F}_p(N) \rightarrow \mathcal{F}_p(L \otimes_R N)$. Now, in order to prove the proposition for the case $\mathcal{F}_p = \otimes^p$, it suffices to show that this last map is just the natural isomorphism that “reorders the factors”. Indeed, let $x_1, \dots, x_n \in L$ and $q = (q_1, \dots, q_n) \in \mathbb{N}^n$ such that $|q| = \sum_i q_i = p$; then β sends the generator $x_1^{[q_1]} \cdot \dots \cdot x_n^{[q_n]}$ to $\binom{p}{q_1, \dots, q_n} \cdot x_1^{\otimes q_1} \otimes \dots \otimes x_n^{\otimes q_n}$. On the other hand, pick any $y \in \otimes^p N$ and let $R[T] = R[T_1, \dots, T_r]$ be the polynomial R -algebra in n variables; write $(T_1 \otimes x_1 + \dots + T_n \otimes x_n)^{\otimes p} \otimes y = \psi_{R[T],y}(T_1 \otimes x_1 + \dots + T_n \otimes x_n) = \sum_{r \in \mathbb{N}^n} T^r \otimes w_r$ with $w_r \in \otimes^p(L \otimes_R N)$. Then $\psi((x_1^{[q_1]} \cdot \dots \cdot x_n^{[q_n]}) \otimes y) = w_q$ (see [21] pp.266-267) and the claim follows easily. Next notice that $\Gamma^p(L)$ is flat, so that tensoring with $\Gamma^p(L)$ commutes with taking coinvariants (resp. invariants) under the action of the symmetric group; this implies the assertion for $\mathcal{F}_p = \text{Sym}^p$ (resp. $\mathcal{F}_p = \text{TS}^p$). To deal with $\mathcal{F}_p = \Lambda^p$ recall that for any V -module M and

$p > 0$ we have the antisymmetrization operator $a_M = \sum_{\sigma \in S_p} \text{sgn}(\sigma) \cdot \sigma : \otimes^p M \rightarrow \otimes^p M$ and a surjection $\Lambda^p(M) \rightarrow \text{Im}(a_M)$ which is an isomorphism for M free, hence for M flat. The result for $\mathcal{F}_p = \otimes^p$ (and again the flatness of $\Gamma^p(L)$) then gives $\Gamma^p(L) \otimes \text{Im}(a_N) \simeq \text{Im}(a_{L \otimes_R N})$, hence the assertion for $\mathcal{F}_p = \Lambda^p$ and N flat. For general N let $F_1 \xrightarrow{\partial} F_0 \xrightarrow{\varepsilon} N \rightarrow 0$ be a presentation with F_i free. Define $j_0, j_1 : F_0 \oplus F_1 \rightarrow F_0$ by $j_0(x, y) = x + \partial(y)$ and $j_1(x, y) = x$. By functoriality we derive an exact sequence

$$\Lambda^p(F_0 \oplus F_1) \rightrightarrows \Lambda^p(F_0) \longrightarrow \Lambda^p(N) \longrightarrow 0$$

which reduces the assertion to the flat case. For $\mathcal{F}_p = \Gamma^p$ the same reduction argument works as well (cf. [21] p.284) and for flat modules the assertion for Γ^p follows from the corresponding assertion for TS^p . \square

Lemma 4.1.10. *Let A be a simplicial almost algebra, L, E and F three A -modules, $f : E \rightarrow F$ a quasi-isomorphism. If L is flat or E, F are flat, then $L \otimes_A f : L \otimes_A E \rightarrow L \otimes_A F$ is a quasi-isomorphism.*

Proof. It is deduced directly from [14] (I.3.3.2.1) by applying $M \mapsto M_l$. \square

As usual, this allows one to show that $\otimes : \text{Hot}_\bullet(A) \times \text{Hot}_\bullet(A) \rightarrow \text{Hot}_\bullet(A)$ admits a left derived functor $\mathbf{L} \otimes : \mathbf{D}_\bullet(A) \times \mathbf{D}_\bullet(A) \rightarrow \mathbf{D}_\bullet(A)$. If R is a simplicial V -algebra then we have essentially commutative diagrams

$$\begin{array}{ccc} \mathbf{D}_\bullet(R) \times \mathbf{D}_\bullet(R) & \xrightarrow{\mathbf{L} \otimes} & \mathbf{D}_\bullet(R) \\ \updownarrow & & \updownarrow \\ \mathbf{D}_\bullet(R^a) \times \mathbf{D}_\bullet(R^a) & \xrightarrow{\mathbf{L} \otimes} & \mathbf{D}_\bullet(R^a) \end{array}$$

where again the downward (resp. upward) functors are induced by localisation (resp. by $M \mapsto M_l$).

We mention the derived functors of the non-additive functor \mathcal{F}_p defined above in the simplest case of modules over a constant simplicial ring. Let A be a (commutative) almost algebra.

Lemma 4.1.11. *If $u : X \rightarrow Y$ is a quasi-isomorphism of flat $s.A$ -modules then $\mathcal{F}_p^a(u) : \mathcal{F}_p^a(X) \rightarrow \mathcal{F}_p^a(Y)$ is a quasi-isomorphism.*

Proof. This is deduced from [14] (I.4.2.2.1) applied to $N(X_l) \rightarrow N(Y_l)$ which is a quasi-isomorphism of chain complexes of flat A_{ll} -modules. We note that *loc. cit.* deals with a more general mixed simplicial construction of \mathcal{F}_p which applies to bounded above complexes, but one can check that it reduces to the simplicial definition for complexes in $\mathcal{C}_\bullet(A_{ll})$. \square

Using the lemma one can construct $L\mathcal{F}_p^a : \mathbf{D}_\bullet(s.A) \rightarrow \mathbf{D}_\bullet(s.A)$. If R is a V -algebra we have the derived category version of the essentially commutative squares (4.1.8), relating $L\mathcal{F}_p : \mathbf{D}_\bullet(s.R) \rightarrow \mathbf{D}_\bullet(s.R)$ and $L\mathcal{F}_p^a : \mathbf{D}_\bullet(s.R^a) \rightarrow \mathbf{D}_\bullet(s.R^a)$.

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REFERENCES

- [1] H.BASS, Algebraic K-theory. W.A. Benjamin; New York-Amsterdam 1968.
- [2] N.BOURBAKI, Algèbre Commutative. Hermann; Paris 1961.

- [3] N.BOURBAKI, Topologie Générale. *Hermann*; Paris 1971.
- [4] P.DELIGNE, J.MILNE, Tannakian categories. *Lect. Notes Math.* 900 Berlin Heidelberg New York: Springer (1982) pp.101-228.
- [5] J.DIEUDONNÉ, A.GROTHENDIECK, Éléments de Géométrie Algébrique - Chapitre II. *Publ. Math. IHES* 8 (1961).
- [6] G.FALTINGS, Almost étale extensions. *Preprint Max-Planck-Institut für Mathematik* 3 (1998).
- [7] D.FERRAND, Descente de la platitude par un homomorphisme fini. *C.R. Acad. Sc. Paris* 269 (1969) pp.946-949.
- [8] P.GABRIEL, Des catégories abéliennes. *Bull. Soc. Math. France* 90 (1962) pp.323-449.
- [9] J.GIRAUD, Cohomologie non abélienne. *Grundl. Math. Wiss.* 179; Berlin Heidelberg New York: Springer (1971).
- [10] A.GROTHENDIECK ET AL., Revêtements Étales et Groupe Fondamental. *Lect. Notes Math.* 224; Berlin Heidelberg New York: Springer (1971).
- [11] A.GROTHENDIECK ET AL., Théorie des Topos et Cohomologie Étale des Schémas, Tome 3. *Lect. Notes Math.* 305; Berlin Heidelberg New York: Springer (1973).
- [12] L.GRUSON, Dimension homologique des modules plats sur un anneau commutatif noetherien. *Symposia Mathematica* Vol. XI; Academic Press, London (1973) pp. 243–254..
- [13] L.GRUSON, M.RAYNAUD, Critères de platitude et de projectivité. *Invent. Math.* 13 (1971) pp.1-89.
- [14] L.ILLUSIE, Complexe cotangent et déformations I. *Lect. Notes Math.* 239; Berlin Heidelberg New York: Springer (1971).
- [15] S.LANG, Algebra - Third edition. *Addison-Wesley* (1993).
- [16] D.LAZARD, Autour de la platitude. *Bull. Soc. Math. France* 97 (1969) pp.81-128.
- [17] S.MAC LANE, Categories for the working mathematician. *Grad. Text Math.* 5; Berlin Heidelberg New York: Springer (1971).
- [18] H.MATSUMURA, Commutative ring theory. *Cambridge Univ. Press* (1986).
- [19] B.MITCHELL, Rings with several objects. *Advances in Math.* 8 (1972) pp.1-161.
- [20] J.-P.OLIVIER, Descente par morphismes purs. *C.R. Acad. Sc. Paris* 271 (1970) pp.821-823.
- [21] N.ROBY, Lois polynômes et lois formelles en théorie des modules. *Ann.Sci.E.N.S.* 80 (1963) pp.213-348.
- [22] C.WEIBEL, An introduction to homological algebra. *Cambridge Univ. Press* (1994).

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